The homomorphism h_* depends not only on the map $h : X \to Y$ but also on the choice of the base point x_0. (Once x_0 is chosen, y_0 is determined by h.) So some notational difficulty will arise if we want to consider several different base points for X. If x_0 and x_1 are two different points of X, we cannot use the same symbol h_* to stand for two different homomorphisms, one having domain $\pi_1(X, x_0)$ and the other having domain $\pi_1(X, x_1)$. Even if X is path connected, so these groups are isomorphic, they are still not the same group. In such a case, we shall use the notation

$$(h_{x_0})_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

for the first homomorphism and $(h_{x_1})_*$ for the second. If there is only one base point under consideration, we shall omit mention of the base point and denote the induced homomorphism merely by h_*.

The induced homomorphism has two properties that are crucial in the applications. They are called its "functorial properties" and are given in the following theorem:

Theorem 52.4. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Proof. The proof is a triviality. By definition,

$$(k \circ h)_*([f]) = [(k \circ h) \circ f],$$
$$(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)].$$

Similarly, $i_*([f]) = [i \circ f] = [f]$. ■

Corollary 52.5. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

Proof. Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h. Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, x_0); and $h_* \circ k_* = (h \circ k)_* = j_*$, where j is the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphisms of the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, k_* is the inverse of h_*. ■

Exercises

1. A subset A of $\mathbb{R}^n$ is said to be ***star convex*** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.
   (a) Find a star convex set that is not convex.
   (b) Show that if A is star convex, A is simply connected.

2. Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

3. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

4. Let $A \subset X$; suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for each $a \in A$. (The map r is called a ***retraction*** of X onto A.) If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \longrightarrow \pi_1(A, a_0)$$

is surjective.

5. Let A be a subspace of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the trivial homomorphism (the homomorphism that maps everything to the identity element).

6. Show that if X is path connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphisms of the groups involved. More precisely, let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}.$$

This equation expresses the fact that the following diagram of maps "commutes."

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \downarrow \hat{\alpha} & & \downarrow \hat{\beta} \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

7. Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Show that this operation makes the set $\Omega(G, x_0)$ into a group.
(b) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.
(c) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [*Hint:* Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]
(d) Show that $\pi_1(G, x_0)$ is abelian.

§53 Covering Spaces

We have shown that any convex subspace of $\mathbb{R}^n$ has a trivial fundamental group; we turn now to the task of computing some fundamental groups that are not trivial. One of the most useful tools for this purpose is the notion of *covering space*, which we introduce in this section. Covering spaces are also important in the study of Riemann surfaces and complex manifolds. (See [A-S].) We shall study them in more detail in Chapter 13.