We have proved the bisection theorem for bounded polygonal regions in the plane. However, all that was needed in the proof was the existence of an additive area function for A_1 and A_2. Thus, the theorem holds for any two sets A_1 and A_2 that are "Jordan-measurable" in the sense used in analysis.

These theorems generalize to higher dimensions, but the proofs are considerably more sophisticated. The generalized version of the bisection theorem states that given n Jordan-measurable sets in $\mathbb{R}^n$, there exists a plane of dimension $n - 1$ that bisects them all. In the case $n = 3$, this result goes by the pleasant name of the "ham sandwich theorem." If one considers a ham sandwich to consist of two pieces of bread and a slab of ham, then the bisection theorem says that one can divide each of them precisely in half with a single whack of a cleaver!

Exercises

1. Prove the following "theorem of meteorology": At any given moment in time, there exists a pair of antipodal points on the surface of the earth at which both the temperature and the barometric pressure are equal.
2. Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective. [*Hint:* If $p \in S^2$, then $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$.]
3. Let $h : S^1 \to S^1$ be continuous and antipode-preserving with $h(b_0) = b_0$. Show that h_* carries a generator of $\pi_1(S^1, b_0)$ to an *odd* power of itself. [*Hint:* If k is the map constructed in the proof of Theorem 57.1, show that k_* does the same.]
4. Suppose you are given the fact that for each n, no continuous antipode-preserving map $h : S^n \to S^n$ is nulhomotopic. (This result can be proved using more advanced techniques of algebraic topology.) Prove the following:
   (a) There is no retraction $r : B^{n+1} \to S^n$.
   (b) There is no continuous antipode-preserving map $g : S^{n+1} \to S^n$.
   (c) (Borsuk-Ulam theorem) Given a continuous map $f : S^{n+1} \to \mathbb{R}^{n+1}$, there is a point x of S^{n+1} such that $f(x) = f(-x)$.
   (d) If $A_1, \ldots, A_{n+1}$ are bounded measurable sets in $\mathbb{R}^{n+1}$, there exists an n-plane in $\mathbb{R}^{n+1}$ that bisects each of them.

§58 Deformation Retracts and Homotopy Type

As we have seen, one way of obtaining information about the fundamental group of a space X is to study the covering spaces of X. Another is one we discuss in this section, which involves the notion of *homotopy type*. It provides a method for reducing the problem of computing the fundamental group of a space to that of computing the fundamental group of some other space—preferably, one that is more familiar.

We begin with a lemma.