Excercises 10

1) Let $\mathbb{R}^m$ and $\mathbb{C}^m$ be equipped with their standard Euclidean metric g given by

$$g(z, w) = \text{Re} \sum_{k=1}^{m} z_k \bar{w}_k$$

and let

$$T^m = \{z \in \mathbb{C}^m | \ |z_1| = ... = |z_m| = 1\}$$

be the m-dimensional torus in $\mathbb{C}^m$ with the induced metric. Find an isometric immersion $\phi : \mathbb{R}^m \to T^m$, determine all geodesics on T^m and prove that the torus is flat.

2)

Let the Lie group $S^3 \cong \mathbf{SU}(2)$ be equipped with the metric

$$g(Z, W) = \frac{1}{2} \text{Re}(\text{trace}(\bar{Z}^t W)).$$

(i) Find an orthonormal basis for $T_e\mathbf{SU}(2)$.
(ii) Prove that $(\mathbf{SU}(2), g)$ has constant sectional curvature $+1$.

3)

Let H^m be the m-dimensional hyperbolic space modelled on the upper half space $\mathbb{R}^+ \times \mathbb{R}^{m-1}$ equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{x_1^2} \langle X, Y \rangle_{\mathbb{R}^m},$$

where $x = (x_1, \ldots, x_m) \in H^m$. For $k = 1, \ldots, m$ let the vector fields $X_k \in C^\infty(TH^m)$ be given by

$$(X_k)_x = x_1 \cdot \frac{\partial}{\partial x_k}$$

and define the operation $*$ on H^m by

$$(\alpha, x) * (\beta, y) = (\alpha \cdot \beta, \alpha \cdot y + x).$$

Prove that

(i) $(H^m, *)$ is a Lie group,
(ii) the vector fields $X_1, \ldots, X_m$ are left-invariant,
(iii) $[X_k, X_l] = 0$ and $[X_1, X_k] = X_k$ for $k, l = 2, ..., m$,
(iv) the metric g is left-invariant,
(v) (H^m, g) has constant curvature -1.