1)

Let M be a smooth manifold and $\hat{\nabla}$ be a connection on the tangent bundle (TM, M, π). Prove that the torsion $T : C_2^\infty(TM) \to C_1^\infty(TM)$ of $\hat{\nabla}$ is a tensor field of type $(2,1)$.

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2)

Let Sol^3 be the 3-dimensional subgroup of $\mathbf{SL}_3(\mathbb{R})$

given by

$$Sol^3 = \{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mid p = (x, y, z) \in \mathbb{R}^3 \}.$$

Let $X, Y, Z \in \mathfrak{g}$ be left-invariant vector fields on Sol^3 such that

$$X_e = \frac{\partial}{\partial x}|_{p=0}, \quad Y_e = \frac{\partial}{\partial y}|_{p=0} \text{ and } Z_e = \frac{\partial}{\partial z}|_{p=0}.$$

Show that

$$[X, Y] = 0, \quad [Z, X] = X \text{ and } [Z, Y] = -Y.$$

Let g be a left-invariant Riemannian metric on G such that $\{X, Y, Z\}$ is an orthonormal basis for the Lie algebra $\mathfrak{g}$. Calculate the vector fields

$$\nabla_X Y, \quad \nabla_Y X, \quad \nabla_X Z, \quad \nabla_Z X, \quad \nabla_Y Z \text{ and } \nabla_Z Y.$$

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3)

Let $\mathbf{SO}(m)$ be the special orthogonal group equipped

with the metric

$$\langle X, Y \rangle = \frac{1}{2}\text{trace}(X^t Y).$$

Prove that $\langle , \rangle$ is left-invariant and that for any left-invariant vector fields $X, Y \in \mathfrak{so}(m)$ we have

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Let A, B, C be elements of the Lie algebra $\mathfrak{so}(3)$ with

$$A_e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B_e = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ C_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Prove that $\{A, B, C\}$ is an orthonormal basis for $\mathfrak{so}(3)$ and calculate

$$\nabla_A B, \quad \nabla_B C \quad \text{and} \quad \nabla_C A.$$

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4)

Let $\mathbf{SL}_2(\mathbb{R})$ be the real special linear group equipped with the metric

$$\langle X, Y \rangle_p = \frac{1}{2} \operatorname{trace}((p^{-1}X)^t (p^{-1}Y)).$$

Let A, B, C be elements of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ with

$$A_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that $\{A, B, C\}$ is an orthonormal basis for $\mathfrak{sl}_2(\mathbb{R})$ and calculate

$$\nabla_A B, \quad \nabla_B C \quad \text{and} \quad \nabla_C A.$$
