1)

Let m be a positive integer and $\phi : \mathbb{R}^m \to \mathbb{C}^m$ be the standard parametrization of the m-dimensional torus T^m in $\mathbb{C}^m$ given by $\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$. Prove that ϕ is an isometric parametrization.

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2)

Let m be a positive integer and

$$\pi_m : (S^m - \{(1, 0, \dots, 0)\}, \langle , \rangle_{\mathbb{R}^{m+1}}) \to (\mathbb{R}^m, \frac{4}{(1 + |x|^2)^2} \langle , \rangle_{\mathbb{R}^m})$$

be the **stereographic projection** given by

$$\pi_m : (x_0, \dots, x_m) \mapsto \frac{1}{1 - x_0}(x_1, \dots, x_m).$$

Prove that π_m is an isometry.

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3)

Let $B_1^2(0)$ be the open unit disk in the complex plane equipped with the hyperbolic metric

$$g(X, Y) = \frac{4}{(1 - |z|^2)^2} \langle X, Y \rangle_{\mathbb{R}^2}.$$

Equip the upper half plane $\{z \in \mathbb{C} |\ \text{Im}(z) > 0\}$ with the Riemannian metric

$$g(X, Y) = \frac{1}{\text{Im}(z)^2} \langle X, Y \rangle_{\mathbb{R}^2}$$

and prove that the holomorphic function

$$\pi : B_1^2(0) \to \{z \in \mathbb{C} |\ \text{Im}(z) > 0\}$$

given by

$$\pi : z \mapsto \frac{i + z}{1 + iz}$$

is an isometry.

4)

Equip the unitary group $\mathbf{U}(m)$ as a submanifold of $\mathbb{C}^{m\times m}$ with the induced metric given by

$$\langle Z, W\rangle = \mathrm{Re}(\mathrm{trace}(\bar{Z}^t W)).$$

Show that for each $p \in \mathbf{U}(m)$ the left translation $L_p : \mathbf{U}(m) \to \mathbf{U}(m)$ given by $L_p : q \mapsto pq$ is an isometry.

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5)

Prove that the antipodal mapping $A: S^n \to S^n$ given by $A(p) = -p$ is an isometry of S^n.

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6)

A function $g: \mathbf{R} \to \mathbf{R}$ given by $g(t) = yt+x, \quad t, x, y \in \mathbf{R}, \quad y > 0$, is called a *proper affine function*. The subset of all such functions with respect to the usual composition law forms a Lie group G. As a differentiable manifold G is simply the upper half-plane $\{(x, y) \in \mathbf{R}^2; y > 0\}$ with the differentiable structure induced from$\mathbf{R}^2$. Prove that:

(a) The left-invariant Riemannian metric of G which at the neutral element $e = (0, 1)$ coincides with the Euclidean metric ($g_{11} = g_{22} = 1$, $g_{12} = 0$) is given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$, (this is the metric of the non-euclidean geometry of Lobatchevski).

(b) Putting $(x, y) = z = x + iy$, $i = \sqrt{-1}$, the transformation $z \to z' = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbf{R}$, $ad - bc = 1$ is an isometry of G.

Hint: Observe that the first fundamental form can be written as:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = -\frac{4dzd\bar{z}}{(z - \bar{z})^2}.$$