Exercises

1. Show that if A is a deformation retract of X, and B is a deformation retract of A, then B is a deformation retract of X.
2. For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine for each space which of the three alternatives holds.
   (a) The "solid torus," $B^2 \times S^1$.
   (b) The torus T with a point removed.
   (c) The cylinder $S^1 \times I$.
   (d) The infinite cylinder $S^1 \times \mathbb{R}$.
   (e) $\mathbb{R}^3$ with the nonnegative x, y, and z axes deleted.
   The following subsets of $\mathbb{R}^2$:
   (f) $\{x \mid \|x\| > 1\}$
   (g) $\{x \mid \|x\| \geq 1\}$
   (h) $\{x \mid \|x\| < 1\}$
   (i) $S^1 \cup (\mathbb{R}_+ \times 0)$
   (j) $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$
   (k) $S^1 \cup (\mathbb{R} \times 0)$
   (l) $\mathbb{R}^2 - (\mathbb{R}_+ \times 0)$
3. Show that given a collection $\mathcal{C}$ of spaces, the relation of homotopy equivalence is an equivalence relation on $\mathcal{C}$.
4. Let X be the figure eight and let Y be the theta space. Describe maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverse to each other.
5. Recall that a space X is said to be *contractible* if the identity map of X to itself is nulhomotopic. Show that X is contractible if and only if X has the homotopy type of a one-point space.
6. Show that a retract of a contractible space is contractible.
7. Let A be a subspace of X; let $j : A \to X$ be the inclusion map, and let $f : X \to A$ be a continuous map. Suppose there is a homotopy $H : X \times I \to X$ between the map $j \circ f$ and the identity map of X.
   (a) Show that if f is a retraction, then j_* is an isomorphism.
   (b) Show that if H maps $A \times I$ into A, then j_* is an isomorphism.
   (c) Give an example in which j_* is not an isomorphism.

***8.** Find a space X and a point x_0 of X such that inclusion $\{x_0\} \to X$ is a homotopy equivalence, but $\{x_0\}$ is not a deformation retract of X. [*Hint:* Let X be the subspace of $\mathbb{R}^2$ that is the union of the line segments $(1/n) \times I$, for $n \in \mathbb{Z}_+$, the line segment $0 \times I$, and the line segment $I \times 0$; let x_0 be the point $(0, 1)$. If $\{x_0\}$ is a deformation retract of X, show that for any neighborhood U of x_0, the path component of U containing x_0 contains a neighborhood of x_0.]

9. We define the *degree* of a continuous map $h : S^1 \to S^1$ as follows:

   Let b_0 be the point $(1, 0)$ of S^1; choose a generator γ for the infinite cyclic group $\pi_1(S^1, b_0)$. If x_0 is any point of S^1, choose a path α in S^1 from b_0 to x_0,