Excercises 6

---

1)

Let the spheres S^1, S^3 and the Lie groups $\mathbf{SO}(n)$, $\mathbf{O}(n)$, $\mathbf{SU}(n)$, $\mathbf{U}(n)$ be equipped with their standard differentiable structures prove the following diffeomorphisms

$$S^1 \cong \mathbf{SO}(2), \quad S^3 \cong \mathbf{SU}(2),$$

$$\mathbf{SO}(n) \times \mathbf{O}(1) \cong \mathbf{O}(n), \quad \mathbf{SU}(n) \times \mathbf{U}(1) \cong \mathbf{U}(n).$$

---

2)

Prove that the matrices

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ \mathbf{1} & 0 \end{pmatrix}$$

form a basis for the tangent space $T_e\mathbf{SL}_2(\mathbb{R})$ of the real special linear group $\mathbf{SL}_2(\mathbb{R})$ at the neutral element e. For each $k = 1, 2, 3$ find an explicit formula for the curve $\gamma_k : \mathbb{R} \to \mathbf{SL}_2(\mathbb{R})$ given by

$$\gamma_k : s \mapsto \mathrm{Exp}(sX_k).$$

---

3)

Suppose that $A = (a^i_j)$ and $B = (b^i_j)$ are $n \times n$ real matrices and

$$X_A = \sum_{i,j,k=1}^{n} a^i_j x^k_i \frac{\partial}{\partial x^k_j}, \quad X_B = \sum_{i,j,k=1}^{n} b^i_j x^k_i \frac{\partial}{\partial x^k_j}$$

are the corresponding left invariant vector fields on $GL(n, \mathbb{R})$. Show that

$$[X_A, X_B] = X_{[A,B]}, \quad \text{where} \quad [A, B] = AB - BA.$$
