(b) Given loops f and g in B, let $\tilde{f}$ and $\tilde{g}$ be liftings of them to E that begin at e_0. Then $\phi([f]) = \tilde{f}(1)$ and $\phi([g]) = \tilde{g}(1)$. We show that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$.

First, suppose that $[f] \in H * [g]$. Then $[f] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Now the product $\tilde{h} * \tilde{g}$ is defined, and it is a lifting of $h * g$. Because $[f] = [h * g]$, the liftings $\tilde{f}$ and $\tilde{h} * \tilde{g}$, which begin at e_0, must end at the same point of E. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E, so that $\phi([f]) = \phi([g])$. See Figure 54.3.

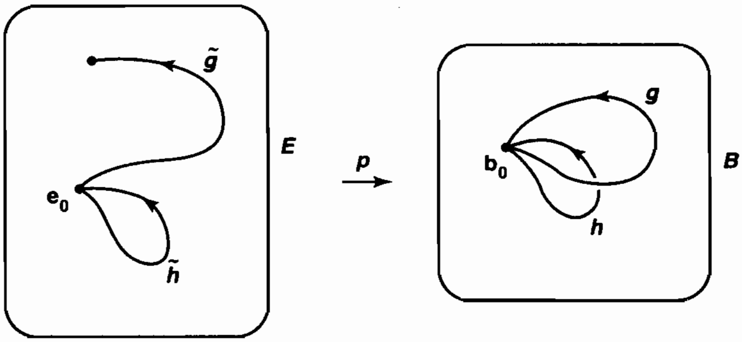

Figure 54.3

Now suppose that $\phi([f]) = \phi([g])$. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E. The product of $\tilde{f}$ and the reverse of $\tilde{g}$ is defined, and it is a loop $\tilde{h}$ in E based at e_0. By direct computation, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy in E between the loops $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy in B between $h * g$ and f, where $h = p \circ \tilde{h}$. Thus $[f] \in H * [g]$, as desired.

If E is path connected, then ϕ is surjective, so that Φ is surjective as well.

(c) Injectivity of Φ means that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$. Applying this result in the case where g is the constant loop, we see that $\phi([f]) = e_0$ if and only if $[f] \in H$. But $\phi([f]) = e_0$ precisely when the lift of f that begins at e_0 also ends at e_0. ∎

Exercises

1. What goes wrong with the "path-lifting lemma" (Lemma 54.1) for the local homeomorphism of Example 2 of §53?
2. In defining the map $\tilde{F}$ in the proof of Lemma 54.2, why were we so careful about the order in which we considered the small rectangles?
3. Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

4. Consider the covering map $p : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 - \mathbf{0}$ of Example 6 of §53. Find liftings of the paths

$$f(t) = (2 - t, 0),$$
$$g(t) = ((1 + t)\cos 2\pi t, (1 + t)\sin 2\pi t)$$
$$h(t) = f * g.$$

Sketch these paths and their liftings.

5. Consider the covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ of Example 4 of §53. Consider the path

$$f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$$

in $S^1 \times S^1$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\tilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$, and sketch it.

6. Consider the maps $g, h : S^1 \to S^1$ given $g(z) = z^n$ and $h(z) = 1/z^n$. (Here we represent S^1 as the set of complex numbers z of absolute value 1.) Compute the induced homomorphisms g_*, h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself. [*Hint:* Recall the equation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.]

7. Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

8. Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply connected, then p is a homeomorphism.

§55 Retractions and Fixed Points

We now prove several classical results of topology that follow from our knowledge of the fundamental group of S^1.

Definition. If $A \subset X$, a ***retraction*** of X onto A is a continuous map $r : X \to A$ such that $r|A$ is the identity map of A. If such a map r exists, we say that A is a ***retract*** of X.

Lemma 55.1. *If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.*

Proof. If $r : X \to A$ is a retraction, then the composite map $r \circ j$ equals the identity map of A. It follows that $r_* \circ j_*$ is the identity map of $\pi_1(A, a)$, so that j_* must be injective. ∎

Theorem 55.2 (No-retraction theorem). *There is no retraction of B^2 onto S^1.*

Proof. If S^1 were a retract of B^2, then the homomorphism induced by inclusion $j : S^1 \to B^2$ would be injective. But the fundamental group of S^1 is nontrivial and the fundamental group of B^2 is trivial. ∎