Algebraic Topology

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Contents

CHAPTER 1

Introduction

1. Introduction

Topology is the study of properties of topological spaces invariant under homeomorphisms. See Section 2 for a precise definition of topological space.

In algebraic topology, one tries to attach algebraic invariants to spaces and to maps of spaces which allow us to use algebra, which is usually simpler, rather than geometry. (But, the underlying motivation is to solve geometric problems.)

A simple example is the use of the Euler characteristic to distinguish closed surfaces. The Euler characteristic is defined as follows. Imagine the surface (say a sphere in $\mathbf{R}^3$) triangulated and let n_0 be the number

of vertices, ... Then $\chi = n_0 - n_1 + n_2$. As the picture indicates, this is 2 for a sphere in $\mathbf{R}^3$ but it is 0 for a torus.

This analysis raises some questions. First, how do we know that the number so obtained does not depend on the way the surface is triangulated? Secondly, how do we know the number is a topological invariant?

Our approach will be to show that for reasonable spaces X, we can attach certain groups $H_n(X)$ (called homology groups), and that invariants b_n (called Betti numbers) associated with these groups can be used in the definition of the Euler characteristic. These groups don't depend on particular triangulations. Also, homeomorphic spaces have isomorphic homology groups, so the Betti numbers and the Euler characteristic are topological invariants. For example, for a 2-sphere, $b_0 = 1, b_1 = 0$, and $b_2 = 1$ and $b_0 - b_1 + b_2 = 2$.

Another more profound application of this concept is the Brouwer Fixed Point Theorem.

Let $D^n = \{x \in \mathbf{R}^n \mid |x| \leq 1\}$, and $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$.

THEOREM 1.1 (Brouwer). *Let $f : D^n \to D^n$ $(n \geq 1)$ be a continuous map. Then f has a fixed point, i.e., $\exists x \in D^n$ such that $f(x) = x$.*

PROOF. (modulo this course) Suppose $f : D^n \to D^n$ does not have a fixed point. Define $r : D^n \to S^{n-1}$ as follows. Extend the ray which starts at $f(x)$ and goes to x until it hits the boundary S^{n-1} of the closed ball D^n. Let that be $r(x)$. Note that this is well defined if there are no fixed points. It is also not hard to prove it is continuous. (Do it!) Finally, it has the property $r(x) = x$ for every point in S^{n-1}. (Such a map of a space into a subspace is called a *retraction*. We shall show using homology groups that such a map can't exist.

Some properties of homology theory that will be proved.

(i) $H_{n-1}(D^n) = 0$.
(ii) $H_{n-1}(S^{n-1}) = \mathbf{Z}$, the infinite cyclic group.
(iii) If $r : X \to Y$ is a continuous map of spaces, then for each k, there are induced group homomorphisms $r_k : H_k(X) \to H_k(Y)$.
(iv) Moreover, these group homomorphisms are consistent with composition of functions, i.e., $(r \circ s)_k = r_k \circ s_k$.
(v) The identity map $X \to X$ induces identity homomorphisms of homology groups.

Let $i : S^{n-1} \to D^n$ be the inclusion map. Then $r \circ i = \mathrm{Id}$. Thus

$$H_{n-1}(S^{n-1}) \to H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$$

is the identity homomorphism of $\mathbf{Z}$ which is inconsistent with the middle group being trivial. $\square$

Note that since there are so many conceivable continuous maps, it is not at all clear (even for $n = 2$) on purely geometric grounds that there can't be a retraction r, although it seems intuitively reasonable that no such map can exist. However, by bringing in the homology groups, we reduce the issue to a question of whether a certain type of homomorphism can exist, and the answer to that question is much simpler, basically because there are many fewer homomorphisms between groups than maps between spaces, so it easier to tell the former apart than the latter.

2. Point Set Topology, Brief Review

A metric space is a set X with a real valued function $d(x, y)$ satisfying

(i) $d(x, y) = 0 \Leftrightarrow x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

Given such a space, one can define the concept of continuous function and a variety of other concepts such as compactness, connectedness, etc.

However, metric spaces are not sufficiently general since even in cases where there may be a metric function $d(x, y)$, it may not be apparent what it is. (Also, there are cases of interest where there is no such function.)

Example. The Klein bottle is often defined by a picture of the following type.

Here the two horizontal edges are *identified* in same the direction and the vertical edges are *identified* in opposite directions. Something homeomorphic to this space may be embedded in an appropriate $\mathbf{R}^n$ so using the metric inherited from that, it can be viewed as a metric space. However, that particular representation is hard to visualize.

We need then some other way to describe spaces without using a metric. We do that by means of open sets. In a metric space X, denote $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$ (open ball centered at x or radius ϵ.) A set is open if every point in the set is the center of some open ball contained in the set. Open sets have the following properties.

(i) $\emptyset$ and X are both open sets.
(ii) Any union whatsoever of open sets is open.
(iii) Any finite intersection of open sets is open.

Then we define a *topology* on a set X to be a collection of subsets (which will be called open sets) satisfying these three axioms. A *topological space* is a set, together with some topology on it. (Note that the same set can have many different topologies placed on it.)

Here is an example of a topological space which is not a metric space. Let $X = \{x, y\}$ be a set with two elements. Let the topology be the collection consisting of the following subsets of X: $\emptyset, \{x\}$, and $\{x, y\} = X$.

This can't be a metric space because it doesn't satisfy the following *Hausdorff separation property*: For any two points $x, y \in X$, there exist non-intersecting open sets U and V containing x and y respectively. (These are usually called open *neighborhoods* of the points.) It is easy to see that any metric space has this property, but clearly X in the above example doesn't have it.

A function $f : X \to Y$ of topological spaces is called continuous if the inverse image $f^{-1}(U)$ of every open set U is open. (Similarly for closed sets.) A continuous function $f : X \to Y$ is called a homeomorphism if there is a continuous function $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are respectively the identity maps of X and Y. This implies f is one-to-one and onto as a map of sets. Conversely, if f is one-to-one and onto, it has a set theoretic inverse $g = f^{-1}$, and f is homeomorphism when this inverse is continuous.

Any subset A of a topological space X becomes a topological space by taking as open sets all intersections of A with open sets of X. Then the inclusion map $i : A \to X$ is continuous.

The collection of all topological spaces and continuous maps of topological spaces forms what is called a category. This means among other things that the composition of two continuous functions is continuous and the identity map of any space is continuous. (Later we shall study the concept of category in more detail.) Another important category is the category of groups and homomorphisms of groups. The homology groups $H_n(X)$ together with induced maps $H_n(f) : H_n(X) \to H_n(Y)$ describe what is called a functor from one category to another. This also is a concept we shall investigate in great detail later.

There are various concepts defined for metric spaces which extend easily to topological spaces since they depend only on the concept open set.

A subset A of a topological space X is called compact if every covering of A by a union of open sets can be reduced to a finite subcovering which also covers A. (Some authors, e.g., Bourbaki, insist that a compact space also be Hausdorff.) Compact subsets of $\mathbf{R}^n$ are exactly the closed bounded subsets. (This is not necessarily true for any metric space.) Here are some other facts about compactness.

A closed subset of a compact space is always compact.

In a Hausdorff space, a compact subset is always closed.

If $f : X \to Y$ is continuous and $A \subset X$ is compact, then $f(A)$ is compact.

A topological space X is called connected if it cannot be decomposed $X = U \cup V$ into a disjoint union ($U \cap V = \emptyset$) of two non-empty open sets.

In $\mathbf{R}$, the connected subspaces are precisely the intervals. (See the exercises.)

If $f : X \to Y$ is continuous, and A is a connected subspace of X, then $f(A)$ is a connected subspace of Y.

A set X may have more than one topology, so it can be the underlying set of more than one topological space. In particular, a set X may always be given the discrete topology in which every set is open. (When is a space with the discrete topology Hausdorff? compact? connected?)

One of the important functors we shall describe is the fundamental group. For this purpose, we need a stronger notion than connectedness called path connectedness. To define this we need a preliminary notion. A path in X is a continuous function $\alpha : [0, 1] \to X$.

A space X is called path connected if any two points x, y may be connected by a path, i.e., there is a path α with $\alpha(0) = x$ and $\alpha(1) = y$.

PROPOSITION 1.2. *A path connected space X is connected.*

PROOF. Let $X = U \cup V$ be a decomposition into disjoint open sets. If neither is empty, pick $x \in U$ and $y \in V$, and pick a path α joining x to y.

Since $[0, 1]$ is connected, so is $A = \mathrm{Im}(\alpha)$. On the other hand, $A = (U \cup A) \cap (V \cup A)$ is a decomposition of A into disjoint open sets of A, and neither is empty, so that is a contradiction. $\square$

A connected space is not necessarily path connected, but a locally path connected space which is connected is path connected.

If X, Y are topological spaces, then the Cartesian product $X \times Y$ (consisting of all pairs (x, y)) is made into a topological space as follows.

If U is open in X and V is open in Y, then $U \times V$ is open in $X \times Y$. Moreover, any union of such 'rectangular sets' is also taken to be open in $X \times Y$, and this gives the collection of all open sets. This is in fact the smallest topology which can be put on $X \times Y$ so that the projections $X \times Y \to X$ and $X \times Y \to Y$ are both continuous. A finite product of topological spaces is made into a topological space is an analogous manner. However, an infinite Cartesian product of topological spaces requires more care. (You should look that up if you don't know it.)

A product of two Hausdorff (respectively compact, connected, or path connected) spaces is Hausdorff (respectively, ...).

An n dimensional manifold is a Hausdorff space X with the property that each point $x \in X$ has an open neighborhood which is homeomorphic to an open ball in $\mathbf{R}^n$. Most of the spaces we are interested in algebraic topology are either manifolds or closely related to manifolds. For example, what we usually think of as a surface in $\mathbf{R}^3$ is a 2-manifold. However, there are 2-manifolds (e.g., the Klein bottle) which can't be embedded in $\mathbf{R}^3$ as subspaces.

CHAPTER 2

Homotopy and the Fundamental Group

1. Homotopy

Denote $I = [0,1]$. Let $f, g : X \to Y$ be maps of topological spaces. A *homotopy* from f to g is a map $H : X \times I \to Y$ such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. Thus, on the 'bottom' edge, H agrees with f and on the 'top' edge it agrees with g. The intermediate maps $H(-,t)$ for $0 < t < 1$ may be thought of a 1-parameter family of maps through which f is continuously deformed into g. We say f is *homotopic* to g ($f \sim g$) if there is a homotopy from f to g.

Example 2.1. Let $X = S^1$ and $Y = S^1 \times I$ (a cylinder of radius 1 and height 1). Define $H(x,t) = (x,t)$. (What are f and g?)

Example 2.2. Let $X = S^1$ and $Y = D^2$. Let f be the inclusion of S^1 in D^2 and let g map S^1 to the center of D^2. Then $H(x,t) = (1-t)x$ defines a homotopy of f to g.

Example 2.3. Let $X = Y = \mathbf{R}^n$. Let f be the identity map, and let g be defined by $g(x) = 0$ for all x. Define $H(x,t) = (1-t)x$. Note that the same argument would work for any point with a slightly different H.

If the identity map of a space is homotopic to a constant map (as in Example 2.3), we say the space is *contractible*.

It is also useful to have a relative version of this definition. Let A be a subspace of X. Suppose $f, g : X \to Y$ agree on A. A homotopy relative to A from f to g is a map as above which satisfies in addition $H(x,t) = f(x) = g(x)$ for all $x \in A$.

By taking $A = \emptyset$, we see that the former concept is a special case of the latter concept.

We say f is homotopic to g relative to A ($f \sim_A g$) if there is a homotopy relative to A from f to g.

Proposition 2.4. *Let X, Y be topological spaces and let A be a subset of X. Then $\sim_A$ is an equivalence relation on the set $Map_A(X,Y)$ of maps from X to Y which agree on A.*

PROOF. For notational convenience, drop the subscript $_A$ from the notation.

(i) Reflexive property $f \sim f$: Define $H(x,t) = f(x)$. This is the composition of f with the projection of $X \times I$ on X. Since it is a composition of two continuous maps, it is continuous.

(ii) Symmetric property $f \sim g \Rightarrow g \sim f$: Suppose $H : X \times I \to Y$ is a homotopy (relative to A) of f to g. Let $H'(x,t) = H(x, 1-t)$. Then $H'(x,0) = H(x,1) = g(x)$ and similarly for $t = 1$. Also, if $H(a,t) = f(a) = g(a)$ for $a \in A$, the same is true for H'. H' is a composition of two continuous maps. What are they?

(iii) Transitive property $f \sim g, g \sim h \Rightarrow f \sim h$: This is somewhat harder. Let $H' : X \times I \to Y$ be a homotopy (relative to A) from f to g, and let H'' be such a homotopy of g to h. Define

$$H(x,t) = \begin{cases} H'(x,2t) & \text{for } 0 \le t \le 1/2 \\ H''(x,2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Note that the definitions agree for $t = 1/2$. We need to show H is continuous.

LEMMA 2.5. *Let $Z = A \cup B$ where A and B are closed subspaces of X. Let $F : Z \to Y$ be a function with the property that its restrictions to A and B are both continuous. Then F is continuous.*

(Note that in the above circumstances, specifying a function on A and on B completely determines the function. The only issue is whether or not it is continuous.)

To derive the Proposition from the Lemma, choose $A = X \times [0, 1/2]$ and $B = X \times [1/2, 1]$ and $F = H$. $\square$

PROOF OF THE LEMMA. Let K be a closed subset of Y. By assumption $A \cap F^{-1}(K)$ is a closed subset of A (in the subspace topology of A) and similarly for $B \cap F^{-1}(K)$. However, it is not hard to see that a subset of a closed subspace A which is closed in the subspace topology is closed in the overlying space Z. Hence. $F^{-1}(K) = (F^{-1} \cap A) \cup (F^{-1} \cap B)$ is a union of two closed subsets of Z, so it is closed. $\square$

This Lemma and its extension to more than two closed subsets will be used repeatedly in what follows.

2. The Fundamental Group

The fundamental group of a space is one of the basic concepts of algebraic topology. For example, you may have encountered the concept 'simply connected space' in the study of line integrals in the plane or in complex function theory. For example, Cauchy's theorem in complex function theory is often stated for simply connected regions in the complex plane $\mathbf{C}$. (An open set in the complex plane $\mathbf{C}$ is simply connected if every simple closed curve may be deformed to (is homotopic to) a point (a constant map).) Cauchy's theorem is not true for non-simply connected regions in $\mathbf{C}$. The fundamental group measures how far a space is from being simply connected.

The fundamental group briefly consists of equivalence classes of homotopic closed paths with the law of composition being following one path by another. However, we want to make this precise in a series of steps. Let X be a topological space. As above, let $I = [0,1]$ and also

denote its boundary by $\dot{I} = \{0,1\}$. Then the set of paths $f : I \to X$ is partitioned into equivalence classes by the relation f is homotopic to g relative to $\dot{I}$. Note that equivalent paths start and end in the same point. Denote by $[f]$ the equivalence class of f.

Let f, g be paths in X such that the initial point $g(0)$ of g is the terminal point $f(1)$ of f. Denote by $f * g$ the path obtained by following the path f by the path g. More formally

$$(f * g)(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq 1/2 \\ g(2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Proposition 2.6. *Suppose $f \sim f'$ and $g \sim g'$, both relative to $\dot{I}$. Suppose also the common terminal point of f, f' is the common initial point of g, g'. Then $f * g \sim f' * g'$ relative to $\dot{I}$.*

Proof. Let F be a homotopy for $f \sim f'$ and G one for $g \sim g'$. Define

$$H(t,s) \begin{cases} = F(2t,s) & \text{for } 0 \leq t \leq 1/2 \\ G(2t-1,s) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

$\square$

This implies that the set of equivalence classes $[f]$ of paths in X has a law of composition which is *sometimes* defined, i.e., $[f] * [g] = [f * g]$ makes sense. We shall show that the set of equivalence classes has identity elements and inverses. For each point $x \in X$, let e_x denote the constant map $I \to X$ such that $e_x(t) = x$ for all $t \in I$.

PROPOSITION 2.7. *Let f be a path in X. Then*

$$e_{f(0)} * f \sim f \qquad f * e_{f(1)} \sim f$$

relative to $\dot{I}$.

Note that this says $[e_{f(0)}]$ is a left identity for $[f]$ and $[e_{f(1)}]$ is a right identity for $[f]$.

PROOF. For $f * e_{f(1)} \sim f$, define

$$H(t, s) = \begin{cases} f(2t/(s+1)) & \text{for } 0 \le t \le (s+1)/2 \\ f(1) & \text{for } (s+1)/2 \le t \le 1. \end{cases}$$

A similar argument works for $e_{f(0)} * f \sim f$. (You should at least draw the appropriate diagram.) □

Note that the idea is first to draw an appropriate diagram and then to determine the formulas by doing the appropriate linear reparameterizations for each s.

PROPOSITION 2.8. *Let f, g, h be paths in X such that $f(1) = g(0), g(1) = h(0)$. Then*

$$f * (g * h) \sim (f * g) * h \qquad \textit{relative to } \dot{I}.$$

Note that this tells us that the law of composition on the equivalence classes is associative when defined.

PROOF. Define

$$H(s,t) = \begin{cases} f(4t/(2-s)) & \text{for } 0 \le t \le (2-s)/4 \\ g(4t+s-2) & \text{for } (2-s)/4 \le t \le (3-s)/4 \\ h((4t+s-3)/(1+s)) & \text{for } (3-s)/4 \le t \le 1 \end{cases}$$

□

For a path f in X define another path f' by $f'(t) = f(1-t)$.

PROPOSITION 2.9. *$f * f' \sim e_{f(1)}$ and $f' * f \sim e_{f(0)}$ relative to $\dot{I}$.*

PROOF. Exercise. □

Note that we have all the elements needed for a group except that the law of composition is not always defined. To actually get a group, choose a point x_0 (called a base point) and let $\pi_1(X, x_0)$ be the set of equivalence paths of all paths which start and end at x_0. Such paths are called *loops*.

This set has a unique identity $[e_{x_0}]$. Also, the law of composition is always defined and satisfies the axioms for a group. $\pi_1(X, x_0)$ is called the *fundamental group* of X with base point x_0. (It is also called the Poincare group since he invented it.) It is also common to use the notation $\pi_1(X, x_0)$ because this group is the first of infinitely many groups π_n called homotopy groups.

EXAMPLE 2.10. Let $X = \{x_0\}$ consist of a single point. The the only path (loop) is the constant map $f : I \to \{x_0\}$. Hence, $\pi_1(X, x_0) = \{1\}$, the trivial group.

EXAMPLE 2.11. Let X be a convex subset of $\mathbf{R}^n$, and let x_0 be a base point in X. Then $\pi_1(X, x_0)$ is trivial also. For, let $f : I \to X$ be a loop based at x_0. from f to the constant loop based at x_0 by

$$H(t, s) = sx_0 + (1-s)f(t).$$

Then clearly, $H(t, 0) = f(t)$ and $H(t, 1) = x_0$, as claimed.

EXAMPLE 2.12. Let x_0 be any point in S^1. Then $\pi_1(S^1, x_0) = \mathbf{Z}$, the infinite cyclic group. This is not particularly easy to prove. We will get to it eventually, but you might think a bit about it now. The map $f : I \to S^1$ defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$ (or in complex notation

$f(t) = e^{2\pi t}$) should be a generator (for basepoint $(1, 0)$), but you might have some trouble even proving that it is not homotopic to a constant map.

Let X be a space and x_0 a base point. It is natural to ask how the fundamental group changes if we change the base point. The answer is quite simple, but there is a twist.

Let x_1 be another base point. *Assume X is path connected.* Then there is a path $f : I \to X$ starting at x_0 and ending at x_1. Let f' denote the reverse path as before. Define a function $\pi_1(X, x_0) \to \pi_1(X, x_1)$ as follows. For a loop g based at x_0, send

$$g \mapsto f * g * f'$$

where the right hand side is a loop based at x_1. Since '$*$' is consistent with homotopies relative to $\dot{I}$, it follows that on equivalence classes of loops, this is a well defined map, so we get a function $\phi_f : \pi_1(X, x_0) \to \pi_1(X, x_1)$. It is also true that this function is a homomorphism of groups. For, if g, h are loops based at x_0, we have

$$f * g * h * f' \sim f * g * e_{x_0} * h * f' \sim (f * g * f') * (f * h * f).$$

(Note this also used the fact that '$*$' is associative up to homotopy—which is what allows us to forget about parentheses.) It follows that the function on equivalence classes is a homomorphism.

It is in fact an isomorphism, the inverse map being provided by

$$h \mapsto f' * h * f \qquad \text{where } [h] \in \pi_1(X, x_1).$$

(You should verify that!)

The twist is that the isomorphism depends on the equivalence class of $[f]$, so different paths from x_0 to x_1 could result in different homomorphisms. Note that even in the special case $x_0 = x_1$, we could choose a path f (which would be a loop based at x_0) which would result in a non-identity isomorphism. Namely, if $[f] = \alpha, [g] = \beta$, the isomorphism is the inner automorphism

$$\beta \mapsto \alpha\beta\alpha^{-1}$$

which will be the identity only in the case α is in the center of $\pi_1(X, x_0)$.

The next question to study is how the fundamental group is affected by maps $f : X \to Y$.

Proposition 2.13. *Let X, Y, and Z be spaces. Let A be a subspace of X. Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be maps. If $f \sim_A f'$ and $g \sim_{f(A)} g'$ then $g \circ f \sim_A g' \circ f'$.*

Note under the hypotheses, $f(A) = f'(A)$.

PROOF. Let $H : X \times I \to Y$ be a homotopy of f to f' relative to A. Then it is easy to see that $g \circ H$ is a homotopy of $g \circ f$ to $g \circ f'$ relative to A. Similarly, if $F : Y \times I \to Z$ is a homotopy of g to g' relative to $f(A) = f'(A)$, then $F \circ (f' \times Id)$ is a homotopy of $g \circ f'$ to $g' \circ f'$ relative to A. Now use transitivity of $\sim_A$. □

Suppose now that $f : X \to Y$ is a map, and $x_0 \in X$. If h is a path in X, $f \circ h$ is a path in Y. Moreover, changing to a homotopic path h' results in a homotopic path $f \circ h'$. Hence, we get a function

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)).$$

(Note what happened to the base point!)

PROPOSITION 2.14. *$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a homomorphism.*

PROOF. Let g, h be paths in X such that $g(1) = h(0)$. It is easy to check from the definition of $*$ that

$$f \circ (g * h) = (f \circ g) * (f \circ h).$$

□

In language we will introduce later, we have defined a *functor* from topological spaces (with base points specified) to groups, i.e., for each pair (X, x_0) we have a group $\pi_1(X, x_0)$ and for each map $X \to Y$ such that $f(x_0) = y_0$, we get a group homomorphism $f_* :: \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Moreover, these induced homomorphisms behave in plausible ways.

PROPOSITION 2.15. (*i*) $\mathrm{Id}_* = \mathrm{Id}$*, i.e., the identity map of a space induces the identity homomorphism of its fundamental group.*

(*ii*) *Let $f : X \to Y$ and $g : Y \to Z$ be maps. Then $(g \circ f)_* = g_* \circ f_*$, i.e., the induced map of fundamental groups is consistent with composition.*

PROOF. (i) is obvious. (ii) is also obvious since

$$(g \circ f)_*([h]) = [(g \circ f) \circ h] = [g \circ (f \circ h)] = g_*([f \circ h]) = g_*(f_*([h])).$$

(Make sure you understand each step.) □

3. Homotopy Equivalence

Let $f, f' : X \to Y$ be homotopic maps. Fix a basepoint $x_0 \in X$. f and f' induce homomorphisms $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f'_* : \pi_1(X, x_0) \to \pi_1(Y, y'_0)$ where $y_0 = f(x_0)$ and $y'_0 = f'(x_0)$. If $y_0 = y'_0$ and the homotopy $f \sim f'$ also sends x_0 to y_0, it is not hard to see that $f_* = f'_*$. We want to be able to say what happens if $y_0 \neq y'_0$.

Let $F : X \times I \to Y$ be a homotopy $f \sim f'$. The function defined by $z(t) = F(x_0, t), 0 \leq t \leq 1$ defines a path in Y from y_0 to y'_0. Then as above, we have an isomorphism $\phi_z : \pi_1(Y, y_0) \to \pi_1(Y, y'_0)$ defined on loops g based at y_0 by

$$g \to \overline{z} * g * z.$$

($\overline{z}$ denotes the reverse path.)

Proposition 2.16. *With the above notation, $\phi_z \circ f_* = f'_*$. i.e., the diagram below commutes.*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[rr, "f'_*"] \arrow[dr, "f_*"'] & & \pi_1(Y, y'_0) \\ & \pi_1(Y, y_0) \arrow[ur, "\phi_z"'] & \end{tikzcd}$$

Proof. Let $h : I \to X$ be a loop in X based at x_0. Consider the map $H : I \times I \to Y$ defined by

$$H(t, s) = F(h(t), s).$$

Let u denote the top edge of $I \times I$ as a path in the square, l the left edge traversed downward, b the bottom edge, and r the right edge traversed upward. Then $H \circ u = f' \circ h$, and $H \circ l = \overline{z}, H \circ u = f \circ h, H \circ r = z$. Since the square is a convex set in $\mathbf{R}^2$, we have seen in an exercise that $u \sim l * b * r$ relative to $\{(0, 1), (1, 1)\}$. It follows that in Y $f' \circ h \sim \overline{z} * (f \circ h) * z$ relative to $y'_0 = H(0, 1) = H(1, 1)$. $\square$

We want to consider homotopic maps of spaces to be in some sense the same map. Similarly, if $f : X \to Y$ is a map, we call $g : Y \to X$ a homotopy inverse if $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$. In this case, we say that f is a *homotopy equivalence*. We also say that X and Y are homotopy equivalent.

EXAMPLE 2.17. A space X is homotopy equivalent to a point if and only if it is contractible. (Exercise: Prove this.)

EXAMPLE 2.18. Let X be any space. Then X and $X \times I$ are homotopy equivalent.

To see this let $f : X \to X \times I$ be the inclusion of X on the 'bottom edge' of $X \times I$, i.e. $f(x) = (x, 0)$, and let g be the projection of $X \times I$ on X. In this case, we have equality $g \circ f = \mathrm{Id}_X$. To see $f \circ g \sim \mathrm{Id}_{X\times I}$, define $H : (X \times I) \times I \to X \times I$ by

$$H(x, s, t) = (x, st).$$

(Check that $H(x, s, 0) = (x, 0) = f(g(x, s))$ and $H(x, s, 1) = \mathrm{Id}(x, s)$.)

PROPOSITION 2.19. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

PROOF. Since $g \circ f \sim \mathrm{Id}_X$, it follows from the basic Proposition proved at the beginning of the section that

$$g_* \circ f_* = (g \circ f)_* = \phi_z \circ \mathrm{Id}_* = \phi_z$$

for an appropriate path z in X, It follows that it is an isomorphism. Similarly, $f_* \circ g_*$ is an isomorphism. It is not hard to see from this that f_* has both left and right inverses (as a map) so it is an isomorphism. □

COROLLARY 2.20. *If X is contractible, then $\pi_1(X, x_0) = \{1\}$ (where x_0 is any base point).*

PROOF. Clear. □

A connected space is called simply connected if $\pi_1(X, x_0) = \{1\}$ for every basepoint x_0. That is equivalent to saying that every loop (basepoint arbitrary) is null-homotopic, i.e., homotopic to a point map. If the space is path connected, then it suffices that $\pi_1(X, x_0)$ is trivial for one base point x_0.

A contractible space is thus simply connected, but the converse is not necessarily true. The primary example is S^n for $n \geq 2$, which is simply connected but *not* contractible. We shall establish both these contentions later.

Recall that a subspace A of a space X is called a retract if the inclusion map $i : A \to X$ has a left inverse $r : X \to A$. It is called a *deformation retract* if in addition r can be chosen so that $i \circ r \sim \mathrm{Id}_X$. Note that since $r \circ i = \mathrm{Id}_A$, it follows that A is homotopically equivalent to X.

Example 2.21. A point $\{x\}$ of X is always a retract of X but will be a deformation retract only if X is contractible. (In this case there is only one possible $r : X \to \{x\}$.)

Example 2.22. We saw in the introduction that S^n is not a retract (so not a deformation retract) of D^{n+1}, but this required the development of homology theory.

Example 2.23. S^n is a deformation retract of $X = \mathbf{R}^{n+1} - \{0\}$ (similarly of $D^{n+1} - \{0\}$.) To see this, choose r to be the map projecting a point of $\mathbf{R}^{n+1}$ from the origin onto S^n. It is clear that r is a retraction, i.e., $r \circ i = \mathrm{Id}$. To see that $i \circ r \sim \mathrm{Id}$, define $H : X \times I \to X$ by

$$H(x, t) = x/(1 - t + t|x|).$$

(Note $|x| \neq (t - 1)/t$ since the right hand side is not positive for $0 \leq t \leq 1$.) Then,

$$\begin{aligned} H(x, 0) &= x \\ H(x, 1) &= x/|x| = i(r(x)). \end{aligned}$$

4. Categories and Functors

We now make precise the ideas we alluded to earlier. A category $\mathcal{C}$ consists of the following. First, we have a collection of objects denoted $\mathrm{Obj}(\mathcal{C})$. In addition, for each ordered pair A, B of objects in $\mathrm{Obj}(\mathcal{C})$, we have a set $\mathrm{Hom}(A, B)$ called morphisms from A to B. (We often write $f : A \to B$ for such a morphism, but this does not imply that f is a function from one set to another, or that A and B are even sets.) We assume that the sets $\mathrm{Hom}(A, B)$ are all disjoint from one another. Moreover, for objects A, B, C in $\mathrm{Obj}(\mathcal{C})$, we assume there is given a law of composition

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

denoted

$$(f, g) \mapsto gf.$$

(Note the reversal of order.) Also, we assume this law of composition is associative when defined, i.e., given $f : A \to B, g : B \to C$, and $h : C \to D$, we have

$$h(gf) = (hg)f.$$

Finally, we assume that for each object A in $\mathrm{Obj}(\mathcal{C})$ there is an element $\mathrm{Id}_A \in \mathrm{Hom}(A, A)$ such that

$$\begin{aligned} \mathrm{Id}_A f &= f \qquad \text{for all } f \in \mathrm{Hom}(X, A) \quad \text{and} \\ f \,\mathrm{Id}_A &= f \qquad \text{for all } f \in \mathrm{Hom}(A, X). \end{aligned}$$

Note the distinction between 'collection' and 'set' in the definition. This is intentional, and is meant to allow for categories the objects of which don't form a 'set' in conventional set theory but something larger. There are subtle logical issues involved here which we will ignore.

Examples 2.24. The most basic category is the category *Sets* of all sets and functions from one set to another.

As mentioned previously, the collection of all spaces and continuous maps of spaces is a category Top. Similarly, for *Gp* the category of groups and homomorphisms of groups or *Ab* the category of abelian groups and homomorphisms.

We have also introduced the collection of all spaces with base point (X, x_0). Morphisms in this category are base point preserving maps, i.e, $f : (X, x_0) \to (Y, y_0)$ is a map $f : X \to Y$ such that $f(x_0) = y_0$.

Finally, we may consider the category in which the objects are topological spaces X, but $\mathrm{Hom}(X, Y)$ consists of homotopy classes of homotopic maps from $X \to Y$. Since composition of maps is consistent with homotopy, this makes sense. We call this the homotopy category $Hpty$.

In general, a morphism $f : X \to Y$ in a category $\mathcal{C}$ is called an isomorphism if there is a morphism $g : Y \to X$ such that $gf = \mathrm{Id}_X, fg = \mathrm{Id}_Y$. The isomorphisms in the category Top are called homeomorphisms. The isomorphisms in the category $Hpty$ are called homotopy equivalences.

Given two categories $\mathcal{C}, \mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ associates to each object A in $\mathrm{Obj}(\mathcal{C})$ an object $F(A)$ in $\mathrm{Obj}(\mathcal{D})$ and to each morphism $f : A \to B$ in $\mathcal{C}$ a morphis.m $F(f) : F(A) \to F(B)$ in $\mathcal{D}$. Moreover, we require that

(1) $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$
(2) If $f : A \to B, g : B \to C$ in $\mathcal{C}$, then $F(gf) = F(g)F(f)$.

Examples 2.25. The fundamental group $\pi_1(X, x_0)$ is a functor from the category of spaces with base points to the category of groups.

If G is a group, the group $G/[G, G]$ is abelian. If $f : G \to H$ is a homomorphism, it induces a homomorphism $\overline{f} : G/[G, G] \to H/[H, H]$. Thus, we have an 'abelianization' functor from the category of groups to the category of abelian groups.

If the objects in a category $\mathcal{C}$ have underlying sets and if morphisms are set maps with some additional properties, we can always define the 'forgetful functor' from $\mathcal{C}$ to *Sets* which just associates to an object the underlying set and to a morphism the underlying function. This works for example for the categories of spaces, spaces with base points, and groups but not for the homotopy category.

5. The fundamental group of S^1

We shall prove

THEOREM 2.26. *Let x_0 be any point of S^1. Then $\pi_1(S^1, x_0) \cong \mathbf{Z}$, the infinite cyclic group.*

The proof is rather involved and requires some discussion of the notion of covering space in the special case of S^1. We shall go into this concept in more detail later.

It is most convenient for our discussion to identify S^1 with the points $z \in \mathbf{C}$ with $|z| = 1$. We shall also take $x_0 = 1$. Then the basic loop which turns out to generate $\pi_1(S^1, x_0)$ is $i : I \to S^1$ defined by $i(t) = e^{2\pi i t}$. (Notice, but don't get excited

about the conflict in notation there!) The map i may be factored through the exponential map $E : \mathbf{R} \to S^1$ given by $E(t) = e^{2\pi i t}$, $-\infty < t < \infty$. That is, let $\tilde{i} : I \to \mathbf{R}$ be the inclusion, so $i = E \circ \tilde{i}$.

$E : \mathbf{R} \to S^1$ is an example of a covering map. Note that it is onto, and the inverse image of any point z consists of all integral translates $t + n$, where $n \in \mathbf{Z}$, and t is any *one* element in $\mathbf{R}$ such that $E(t) = z$. In particular, the inverse image of any point is a *discrete* subspace of $\mathbf{R}$.

Note also that while E certainly isn't invertible, it does have an inverse if we restrict to an appropriate subset of S^1. In particular, we may define a logarithm function $L : S^1 - \{-1\} \to \mathbf{R}$ by taking $L(z)$ to be the unique number $t \in (-1/2, 1/2)$ such that $E(t) = e^{2\pi i t} = z$. Of course, there are other possible ranges of the logarithm function, so there are other possible inverses on $S^1 - \{-1\}$. Any one will be completely determined by specifying the image of 1. For the choice we made $L(1) = 0$.

We now want to do something analogous with an arbitrary loop $h : I \to S^1$. First, we prove

LEMMA 2.27 (Lifting Lemma). *Let $h : I \to S^1$ be a path such that $h(0) = 1$, and let $n \in \mathbf{Z}$. Then there exists a unique map $\tilde{h} : I \to \mathbf{R}$ such that*

(i) $h = E \circ \tilde{h}$
(ii) $\tilde{h}(0) = n$.

PROOF. First we show that such a map is unique. Let $\tilde{h}'$ be another such map. Then since $E(\tilde{h}(t)) = E(\tilde{h}'(t))$, it follows from the properties of the exponential function that $E(\tilde{h}(t) - \tilde{h}'(t)) = 1$ for all $t \in I$. Hence, $\tilde{h}(t) - \tilde{h}'(t) \in \mathbf{Z}$ for all $t \in I$. However, $\tilde{h} - \tilde{h}'$ is continuous, so its image is connected. Since it is contained in $\mathbf{Z}$, a discrete subspace of $\mathbf{R}$, it is constant. Since $\tilde{h}(0) = \tilde{h}'(0)$, it follows that $\tilde{h}(t) = \tilde{h}'(t)$ for all $t \in I$.

It is harder to show that $\tilde{h}$ exists. The idea is to break I up into subintervals whose images may be mapped by the logarithm function L to $\mathbf{R}$, and then piece together the results in $\mathbf{R}$. Since h is continuous on a compact set, it is uniformly continuous. That means we can choose $\delta > 0$ such that $|h(t') - h(t'')| < 2$ whenever $|t' - t''| < \delta$. Thus, $h(t')$ and $h(t'')$ will not be antipodal, and $h(t')h(t'')^{-1} \neq -1$.

Fix $0 < t \leq 1$. Choose N such that each subinterval in the partition $0 = t_0 < t_1 = 1/N < t_2 = 2t/N < \cdots < t_N = t$ of length smaller than δ for any $t \leq 1$. Let $g_k : [t_{k-1}, t_k] = I_k \to S^1$ be defined by $g_k(u) = h(u)h(t_{k-1})^{-1}$. Then $h(I_k) \subseteq S^1 - -1$. Hence, we may define $\tilde{g}_k : I_k \to \mathbf{R}$ by $\tilde{g}_k = L \circ g_k$. Define $\tilde{h} : I \to \mathbf{R}$ by

$$\tilde{h}(t) = n + \tilde{g}_1(t_1) + \tilde{g}_2(t_2) + \cdots + \tilde{g}_n(t_n). \qquad \text{Recall } t_n = t.$$

We leave it to the student to prove that $\tilde{h}$ so defined is continuous. Clearly, $\tilde{h}(0) = n$. Also,

$$E(\tilde{h}(t)) = E(n)E(L(g_1(t_1))E(L(g_1(t_2)) \ldots E(L(g_n(t_n)) = \cdots = h(t).$$

Hence, $\tilde{h}$ has the desired properties. $\square$

LEMMA 2.28 (Homotopy Lifting Lemma). *Let $h, h' : I \to S^1$ be homotopic (relative to $\dot{I}$) starting at 1 and ending at the same point.*

Let $H : I \times I \to S^1$ be a homotopy of h to h' relative to $\dot{I}$. Let $\tilde{h}$ and $\tilde{h}'$ be liftings of $\tilde{h}$ and $\tilde{h}'$ respectively such that $\tilde{h}(0) = \tilde{h}'(0)$. Then there is a homotopy $\tilde{H} : I \times I \to \mathbf{R}$ of $\tilde{h}$ to $\tilde{h}'$ relative to $\dot{I}$ such that $H = E \circ \tilde{H}$ and $\tilde{H}(0,0) = \tilde{h}(0)$. In particular, it follows that $\tilde{h}(1) = \tilde{h}'(1)$, so that $\tilde{h}$ and $\tilde{h}'$ start and end at the same points.

Proof. The proof of the existence of $\tilde{H}$ is essentially the same as that of the existence of $\tilde{h}$. (Instead of partitioning the interval $[0, t]$, partition the line segment from $(0, 0)$ to $(t, s) \in I \times I$.

To see that $\tilde{H}$ is constant on $\{0\} \times I$ and $\{1\} \times I$, note that the images in $\mathbf{R}$ of both these line segments are in $\mathbf{Z}$, so by the above discreteness argument, H is constant on those segments. On the bottom edge, $\tilde{H}(-, 0)$ lifts $H(-, 0) = h$ and agrees with $\tilde{h}$ at its left endpoint, so it is $\tilde{h}$. Similarly, for $\tilde{H}(-, 1)$ and $\tilde{h}'$. Hence, $\tilde{H}$ is the desired homotopy. □

Notes 1. (1) Note that the arguments work just as well if h (and h') start at $z_0 \neq 1$. However, then the initial value $\tilde{h}(0)$ must be chosen in the discrete set $E^{-1}(z_0)$.

(2) Both lemmas may be subsumed in a single lemma in which I or $I \times I$ is replaced by any compact convex set in some $\mathbf{R}^n$. Then the initial point can be any point in that set instead of 0 or $(0, 0)$.

We are now in a position to show $\pi_1(S^1, 1) \cong \mathbf{Z}$. First define a function $q : \mathbf{Z} \to \pi_1(S^1, 1)$ by $q(n) = [i]^n$, where $i : I \to S^1$ is the basic loop described above. Clearly, q is a group homomorphism. We may also define a function $p : \pi_1(S^1, 1) \to \mathbf{Z}$ which turns out to be the inverse of q. Let h be a loop in S^1 representing $\alpha \in \pi_1(S^1, 1)$, Let $\tilde{h}$ be the unique lifting of h such that $\tilde{h}(0) = 0$. Then, let $p(\alpha)$ be the other endpoint of $\tilde{h}$ as a path in $\mathbf{R}$, i.e.

$$p([h]) = h(1).$$

By the homotopy covering lemma, $h(1)$ depends only on the equivalence class of h, so p is well defined. It is called the *degree* of h (or of α). The reason for this terminology is clear if you consider $i^{(n)} = i * i * \cdots * i$ (n times) which (for $n > 0$) represents $[i]^n$. The unique lifting $\widetilde{i^{(n)}}$ such that $\tilde{i}^{(n)}(0) = 0$ is given by

$$\tilde{i}^{(n)}(t) = n\tilde{i}(t).$$

(Check this!) Hence, the degree of $[i]^n$ is n. Thus, in this case, the degree counts the number of times the loop goes around S^1, and this should be the interpretation in general. Note that the above argument proves that $p \circ q = \mathrm{Id}_{\mathbf{Z}}$. (It works for $n > 0$. What if $n \leq 0$?) To complete the proof, we need only show that p is one-to-one. To this end, let g and h be loops in S^1 based at 1 and cover them by $\tilde{g}$ and $\tilde{h}$ which both map 0 to 0. If g and h have the same degree, $\tilde{g}$ and $\tilde{h}$ both map 1 to the same point in $\mathbf{R}$. Since $g = E \circ \tilde{g}$ and $h = E \circ \tilde{h}$, it suffices to show that $\tilde{g}$ and $\tilde{h}$ are homotopic relative to $\dot{I}$. This follows from the following Lemma, which we leave an exercise for you.

LEMMA 2.29. *Let h and g be paths in a simply connected space with the same endpoints. Then $h \sim g$ relative to $\dot{I}$.*

Note that since p is the inverse of q, it is an iso-*morphism*. That is,

$$\deg(h * g) = \deg(h) + \deg(g).$$

You might also think about what this means geometrically.

6. Some Applications

We may now use the fundamental group to derive some interesting theorems.

THEOREM 2.30 (Fundamental Theorem of Algebra). *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be a polynomial with complex coefficients. Then $f(z)$ has at least one complex root.*

It follows by high school algebra that it has exactly n roots, counting multiplicities. This theorem was first proved rigorously by Gauss who liked it so much that he gave something like 8 proofs of it during his lifetime. There are proofs which are essentially algebraic, but we can use the fundamental proof to give a proof.

PROOF. We may assume that a_n is not zero for otherwise $z = 0$ is obviously a root. Define

$$F(z,t) = z^n + t(a_1 z^{n-1} + \cdots + a_n) \qquad \text{for} z \in \mathbf{C}, 0 \leq t \leq 1.$$

F defines a homotopy from $f : \mathbf{C} \to \mathbf{C}$ to the nth power function. Restrict z to the circle C_r defined by $|z| = r$. If r is sufficiently large, $F(z,t)$ is never zero. For,

$$\begin{aligned}|z^n + t(a_1 z^{n-1} + \cdots + a_n)| &> |z|^n - |t|(|a_1||z|^{n-1} + \cdots + |a_n|) \\ &= r^n(1 - |t|(|a_1|/r + \ldots |a_n|/r^n),\end{aligned}$$

and if r is sufficiently large, we can make the expression in parentheses smaller than $1/2$. Thus, $F(z,t)$ provides a homotopy of maps from $C_r \to \mathbf{C} - \{0\}$. $F(z,0) = z^n$ and $F(z,1) = f(z)$.

Similarly, define

$$G(z,t) = f(tz).$$

If we assume that $f(z)$ *never vanishes* for $z \in \mathbf{C}$, then this also provides a homotopy for maps $C_r \to \mathbf{C} - \{0\}$. Also, $G(z,0) = a_n$ and $G(z,1) = f(z)$. It follows that the nth power map $p_n : C_r \to \mathbf{C} - \{0\}$ is homotopic to the constant map. Hence, there is a commutative diagram

$$\begin{tikzcd} \pi_1(C_r, r) \arrow[rr, "p_{n*}"] \arrow[dr, "a_{n*}"'] & & \pi_1(\mathbf{C} - \{0\}, r^n) \arrow[dl, "\phi"] \\ & \pi_1(\mathbf{C} - \{0\}, a_n) & \end{tikzcd}$$

for an appropriate isomorphism ϕ. The homomorphism induced by the constant map is trivial. On the other hand, it is not hard to see that $p_n([i])$ (where $i : I \to C_r$ is a generating loop) is non-trivial. (It just wraps around the circle C_{r^n} n times.) This is a contradiction. $\square$

We can now prove a special case of the Brouwer Fixed Point Theorem.

THEOREM 2.31. *For $n = 2$, any continuous map $f : D^n \to D^n$ has a fixed point.*

PROOF. Check the introduction. As there, we may reduce to showing there is no retraction $r : D^2 \to S^1$. ($r \circ i = \mathrm{Id}$ where $i : S^1 \to D^2$ is the inclusion map.) However, any such retraction would

$$r_* \circ i_* = \mathrm{Id}$$

which is not consistent with $\pi_1(D^2, x_0) = \{1\}$, $\pi_1(S_1, x_0)$ not trivial. $\square$

We are also ready to prove a special case of another well known theorem.

THEOREM 2.32. *For $n = 2$, there is no map $f : S^n \to S^{n-1}$ which sends antipodal points to antipodal points; i.e., so that $f(-x) = -f(x)$.*

COROLLARY 2.33. *For $n = 2$, given a map $g : S^n \to \mathbf{R}^n$, there is a point $x \in S^n$ such that $g(-x) = g(x)$.*

A consequence of Corollary 2.33 is that, assuming pressure and temperature vary continuously on the surface of the Earth, there are two antipodal points where the pressure and temperature are simultaneously the same.

Proof of Corollary 2.33. Consider $g(x)-g(-x)$. If the Corollary is false, this never vanishes, so we may define a map $f : S^n \to S^{n-1}$ by

$$f(x) = \frac{g(x) - g(-x)}{|g(x) - g(-x)|}.$$

This map satisfies $f(-x) = -f(x)$. Hence, there is no such map and the Corollary is true. $\square$

Proof of Theorem 2.32. Consider the restriction of f to the upper hemisphere, i.e., the set of $x \in S^2$ with $x_3 \geq 0$. We can map D^2

onto the upper hemisphere of S^2 by projecting upward and then follow this by f. Call the resulting map $\overline{f} : D^2 \to S^1$. $\overline{f}$ on the boundary S^1 of D^2 provides a map $f' : S^1 \to S^1$ which such that $f'(-x) = -f'(x)$. From the diagram

$$\begin{tikzcd} S^1 \arrow[rr] \arrow[dr] & & D^2 \arrow[dl, "\overline{f}"] \\ & S^1 & \end{tikzcd}$$

we see that the homomorphism $\pi_1(S^1, 1) \to \pi_1(S^1, f'(1))$ is trivial (where we view S^1 as imbedded in $\mathbf{C}$ as before). Let $i : I \to S^1$ be defined by $i(t) = e^{2\pi i t}$ as before, and let $h = f' \circ i$. We shall show that $[h] = f'_*([i])$ is nontrvial, thus deriving a contradition. To see this first note that, because of the antipode preserving property of f', we have $h(t) = -h(t - 1/2)$ for $1/2 \leq t \leq 1$.

Lemma 2.34. *Let $h : I \to S^1$ satisfy $h(t) = -h(t - 1/2)$ for $1/2 \leq t \leq 1$. Then* $\deg h$ *is odd. (In particular, it is not zero.)*

Proof. By a suitable rotation of S^1 we may assume $h(0) = 1$ without affecting the argument. Using the lifting lemma, choose $\tilde{h} : [0, 1/2] \to \mathbf{R}$ such that $h(t) = E(\tilde{h}(t))$ and $\tilde{h}(0) = 0$. Since $E(\tilde{h}(1/2)) =$

$h(1/2) = -h(0) = -1$, it follows that $\tilde{h}(1/2) = k+1/2$ for some integer k. Now extend $\tilde{h}$ to I by defining

$$\tilde{h}(t) = k + 1/2 + \tilde{h}(t - 1/2) \qquad \text{for } 1/2 \leq t \leq 1.$$

Note that according to this formula, we get the same value for $t = 1/2$ as before. Also,

$$E(\tilde{h}(t)) = E(k+1/2)E(\tilde{h}(t-1/2)) = -h(t-1/2) = h(t) \qquad \text{for } 1/2 \leq t \leq 1.$$

Finally,

$$\tilde{h}(1) = k + 1/2 + \tilde{h}(1/2) = 2(k + 1/2) = 2k + 1$$

as claimed. □

□

It is intuitively clear that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$ for $n \neq m$, but it is surprisingly difficult to prove. We shall provide a proof now that this is so if $m = 2$ and $n > 2$. (You should think about how to deal with the case $\mathbf{R}^1$ and $\mathbf{R}^2$ yourself.) If R^2 were homeomorphic to $\mathbf{R}^n$, we could asssume there was a homomorphism that sends 0 to 0. (Why?) This would induce in turn a homeomorphism from $\mathbf{R}^2 - \{0\}$ to $\mathbf{R}^n - \{0\}$. The former space is homotopically equivalent to S^1 and the latter to S^{n-1}. Hence, it suffices to prove that S^1 does not have the same homotopy type as S^n for $n > 1$. This follows from the fact that the former is not simply connected while the latter is. We now shall provide a proof that

Theorem 2.35. *S^n is simply connected for $n > 1$.*

Proof. The idea is to write $X = S^n = U \cup V$ where U and V are simply connected open subspaces.

Let $h : I \to X$ be a loop based at $x_0 \in S^n$ whose image lies entirely within U. Let $h' : I \to U$ be the map describing this. Then there is a homotopy $H' : I \times I \to U$ from h' to the constant map at x_0. Following this by the includsion $U \to X$ yields a homotopy into X. Similarly for a loop entirely contained in V. Of course, it is not true that any element of $\pi_1(X, x_0)$ is represented either by a loop entirely in U or one entirely in V, but we shall show below that *in appropriate circumstances* any element is represented by a *product of such loops.*

To write $S^n = U \cup V$ as above, proceed as follows. Let $U = S^n - \{P\}$ and $V = \{P'\}$ where $P = (0, 0, \dots, 1), P' = (0, 0, \dots, -1) \in \mathbf{R}^{n+1}$ are the north and south poles of the sphere. These sets are certainly open. Also, the subspace obtained by deleting an single point Q from S^n is homeomorphic to $\mathbf{R}^n$, so it is simply connected. (By a linear isomorphism, you may assume the point is the north pole. Then, a homeomorphism is provided by stereographic projection from Q which maps $S^n - \{Q\}$ onto the equatorial hyperplane defined by $x_{n+1} = 0$, which may be identified with $\mathbf{R}^n$. It is clear that stereographic projection is one-to-one and onto, and by some simple algebra, one can derive formulas for the transformation which show that it is continuous.)

Thus to prove the theorem, we need only prove the following result.

PROPOSITION 2.36. *Let X be a compact metric space, and suppose $X = U \cup V$ where U and V are open subsets. Suppose $U \cap V$ is connected and $x_0 \in U \cup V$. Then any loop h based at x_0 can be expressed*

$$h \sim_{\dot{I}} h_1 * h_2 * \cdots * h_k$$

where each h_i is either a loop in U or a loop in V.

Note in the application to S^n, the set $U \cup V$ is path connected.

In order to prove Proposition 2.36 we need the following lemma.

LEMMA 2.37 (Lebesgue Covering Lemma). *Let X be a compact metric space, and suppose $X = \bigcup_i U_i$ by open sets. Then there exists $\epsilon > 0$ such that for each $x \in X$, the open ball $B_\epsilon(x)$ (centered at x and of radius ϵ) is contained in U_i for some i (depending in general on x.)*

The number ϵ is called a *Lebesgue number* for the covering.

PROOF. If X is one of the sets in the covering, we are done using any $\epsilon > 0$. Suppose then that all the U_i are proper open sets. Since X is compact, we may suppose the covering is finite consisting of $U_1, \dots, U_n$. For each i define a function $\delta_i : X \to \mathbf{R}$ by

$$\delta_i(x) = \min_{y \notin U_i} d(x, y).$$

($\delta_i(x)$ is the distance of x to the complement of U_i. It is well defined because $X - U_i$ is closed and hence compact.) We leave it to the student to show that δ_i is continuous. Note also that

$$\begin{aligned} \delta_i(x) &> 0 \qquad \text{if } x \in U_i \\ \delta_i(x) &= 0 \qquad \text{if } x \notin U_i. \end{aligned}$$

Define $\delta : X \to \mathbf{R}$ by

$$\delta(x) = \max_i \delta_i(x).$$

(Why is this defined and continuous?) Note that $\delta(x) > 0$, also because there are only finitely many i. Choose ϵ between 0 and the minimum value of $\delta(x)$. (The minimum exists and is positive since X is compact.) Then for each $x \in X$,

$$d(x, z) < \epsilon \Rightarrow d(x, z) < \delta(x) \Rightarrow d(x, z) < \delta_i(x) \quad \text{for some } i.$$

The last statement implies that $z \in U_i$. Thus $B_\epsilon(x) \subseteq U_i$. □

Proof of Proposition 2.36. Let $h : I \to X$ be a path based at x_0 as in the statement of the proposition. If the image of h is entirely contained either in U or in V, we are done. so, assume otherwise. Apply the Lemma to the covering $I = h^{-1}(U) \cup h^{-1}(V)$. Choose a partition $0 = t_0 < t_1 < \dots < t_n = 1$ of I such that each subinterval is of length less than a Lebesgue number ϵ of the covering.

Then the kth subinterval $I_k = [t_{k-1}, t_k]$ either is contained in $h^{-1}(U)$ or in $h^{-1}(V)$. This says the image of the restriction $h'_k : I_k \to X$ is either contained in U or in V. Moreover, by combining intervals and renumbering where necessary, we may assume that if the kth image is in one subset, the $k + 1$st image is in the other set. Let $h_k : I \to X$ be the kth restriction reparameterized so its domain is I. Then

$$h_k \sim_{\dot{I}} h_1 * h_2 * \dots * h_n.$$

We are not quite done, however, since the h_k are not necessarily loops based at x_0. We may remedy this situation as follows. The point

$h(t_k) = h_k(t_k) = h_{k+1}(t_k)$ (for $0 < k < n$) is in both U and V by construction. Since $U \cap V$ is connected (hence, in this case also path connected), we can find a path p_k from x_0 to $h(t_k)$. Then $h_k \sim_{\dot{I}} h_k * \overline{p}_k * p_k$. Hence,

$$h \sim_{\dot{I}} (h_1 * \overline{p}_1) * (p_1 * h_2 * \overline{p}_2) * \cdots * (p_{n-1} * h_n)$$

and the constituents $\overline{p}_{k-1} * h_k * p_k$ on the right are loops based at x_0, each of which is either contained in U or contained in V. $\square$

$\square$

CHAPTER 3

Quotient Spaces and Covering Spaces

1. The Quotient Topology

Let X be a topological space, and suppose $x \sim y$ denotes an equivalence relation defined on X. Denote by $\hat{X} = X/\sim$ the set of equivalence classes of the relation, and let $p : X \to \hat{X}$ be the map which associates to $x \in X$ its equivalence class. We define a topology on $\hat{X}$ by taking as open all sets $\hat{U}$ such that $p^{-1}(\hat{U})$ is open in X. (It is left to the student to check that this defines a topology.) $\hat{X}$ with this topology is called the *quotient space* of the relation.

Example 3.1. Let $X = I$ and define $\sim$ by $0 \sim 1$ and otherwise every point is equivalent just to itself. I leave it to the student to check that $I/\sim$ is homeomorphic to S^1.

Example 3.2. In the diagram below, the points on the top edge of the unit square are equivalent in the indicated direction to the corresponding points on the bottom edge and similarly for the right and left edges.

The resulting space is homeomorphic to the two dimensional torus $S^1 \times S^1$. To see this, map $I \times I \to S^1 \times S^1$ by $(t, s) \to (e^{2\pi it}, e^{2\pi is})$ where $0 \le t, s \le 1$. It is clear that points in $I \times I$ get mapped to the same point of the torus if and only they are equivalent. Hence, we get an induced one-to-one map of the quotient space onto the torus. I leave it to you to check that this map is continuous and that is inverse is continuous.

Example 3.3. In the diagram below, the points on opposite edges are equivalent in pairs in the indicated directions. As mentioned earlier, the resulting quotient space is homeomorphic to the so-called Klein bottle. (For our purposes, we may take that quotient space to be the definition of the Klein bottle.)

An equivalence relation may be specified by giving a partition of the set into pairwise disjoint sets, which are supposed to be the equivalence classes of the relation. One way to do this is to give an onto map $f : X \to Y$ and take as equivalence classes the sets $f^{-1}(y)$ for $y \in Y$. In this case, there will be a bijection $\hat{f} : \hat{X} \to Y$, and it is not hard to see that $\hat{f}$ will be continuous. However, its inverse need not be continuous, i.e., $\hat{X}$ could have 'fewer' open sets than Y. (Can you invent an example?) However, the map $\hat{f}$ will be *bicontinuous* if it is an open (similarly closed) map. In this case, we shall call the map $f : X \to Y$ a *quotient map.*

PROPOSITION 3.4. *Let $f : X \to Y$ be an onto map and suppose X is endowed with an equivalence relation for which the equivalence classes are the set $f^{-1}(y), y \in Y$. If f is an open (closed) map, then f is a quotient map.*

(However, the converse is not true, e.g., the map $X \to \hat{X}$ need not in general be an open map.)

PROOF. If $\hat{U}$ is open (closed) in $\hat{X}$, then $p^{-1}(\hat{U})$ is open (closed) in X, and

$$\hat{f}(\hat{U}) = \hat{f}(p((p^{-1}(\hat{U})))) = f(p^{-1}(\hat{U}))$$

is open (closed) in Y. □

The above discussion is a special case of the following more general *universal mapping property* of quotient spaces.

PROPOSITION 3.5. *Let X be a space with an equivalence relation $\sim$, and let $p : X \to \hat{X}$ be the map onto its quotient space. Given any map $f : X \to Y$ such that $x \sim y \Rightarrow f(x) = f(y)$, there exists a unique map $\hat{f} : \hat{X} \to Y$ such that $f = \hat{f} \circ p$.*

PROOF. Define $\hat{f}(\hat{x}) = f(x)$. It is clear that this is defined and that $\hat{f} \circ p = f$. It is also clear that this is the only such function. To see that $\hat{f}$ is continuous, let U be open in Y. Then $f^{-1}(U)$ is open in X. But, by the definition of $\hat{f}$, $p^{-1}(\hat{f}^{-1}(U)) = f^{-1}(U)$, so $\hat{f}^{-1}(U)$ is open in $\hat{X}$. □

It makes it easier to identify a quotient space if we can relate it to a quotient map.

PROPOSITION 3.6. *Let $f : X \to Y$ be a map from from a compact space onto a Hausdorff space. Then f is a quotient map.*

(Note how this could have been used to show that the square with opposite edges identified is homeomorphic to a torus. Since the square is compact and the torus is Hausdorff, all you have to check is that the equivalence relation has equivalence classes the inverse images of points in the torus.)

PROOF. f is a closed map. For, if E is a closed subset of X, then it is compact. Hence, $f(E)$ is compact, and since Y is Hausdorff, it is closed. □

Let X be a space, and let A be a subspace. Define an equivalence relation on X by letting all points in A be equivalent and let any other point be equivalent only to itself. Denote by X/A the resulting quotient space.

EXAMPLE 3.7. Let $X = D^n$ and $A = S^{n-1}$ for $n \geq 1$. There is a map of D^n onto S^n which is one-to-one on the interior and which maps S^{n-1} to a point. (What is it? Try it first for $n = 2$.) It follows from the proposition that D^n/S^{n-1} is homeomorphic to S^n.

Quotient spaces may behave in unexpected ways. For example, the quotient space of a Hausdorff space need not be Hausdorf.

EXAMPLE 3.8. Let $X = I = [0, 1]$ and let $A = (0, 1)$. Let the equivalence classes be $\{0\}$, A, and $\{1\}$. Then $\hat{X}$ has three points: $\hat{0}, \hat{0.5}$, and $\hat{1}$. However, the open sets are

$$\emptyset, \{\hat{0.5}\}, \{\hat{0}, \hat{0.5}\}, \{\hat{0.5}, \hat{1}\}, \hat{X}.$$

Clearly, the problem in this example is connected to the fact that the set A is not closed.

This can't happen in certain reasonable circumstances.

PROPOSITION 3.9. *Suppose $f : X \to Y$ is a quotient map with X compact Hausdorff and f a closed map. Then Y is (compact and) Hausdorff.*

PROOF. Any singleton sets in X are closed since X is Hausdorff. Since f is closed and onto, it follows that singleton sets in Y are also closed. Choose $y, z \neq y \in Y$. Let $E = f^{-1}(y)$ and $F = f^{-1}(z)$.

Let $p \in E$. For every point $q \in F$, we can find an open neighborhood $U(q)$ of p and and open neighborhood $V(q)$ of q which don't intersect. Since F is closed, it is compact, so we can cover F with finitely many such $V(q_i), i = 1 \dots n$. Let $V_p = \cup_{i=1}^n V(q_i)$ and $U_p = \cap_{i=1}^n U(q_i)$. Then $F \subseteq V_p$ and U_p is an open neighborhood of p which is disjoint from V_p. But the collection of open sets $U_p, p \in E$ cover E, so we can pick out a finite subset such that $E \subseteq U = \cup_{j=1}^k U_{p_j}$ and U is disjoint from $V = \cap_{j=1}^k V_{p_j}$ which is still an open set containing F. We have now found disjoint open sets $U \supseteq E$ and $V \supseteq F$. Consider $f(X - U)$ and $f(X - V)$. These are closed sets in Y since f is closed. Hence, $Y - f(X - U)$ and $Y - f(X - V)$ are open sets in Y. However, $y \notin f(X - U)$ since otherwise, $y = f(w)$ with $w \notin U$, which contradicts $f^{-1}(y) \subseteq U$. Hence, $y \in Y - f(X - U)$ and similarly, $z \in Y - f(X - V)$. Thus we need only show that these two open sets in Y are disjoint. But

$$\begin{aligned}
(Y - f(X - U)) \cap (Y - f(X - V)) &= Y - (f(X - U) \cup f(X - V)) \\
&= Y - f((X - U) \cup (X - V)) \\
&= Y - f(X - (U \cap V)) = Y - f(X) \\
&= \emptyset.
\end{aligned}$$

□

A common application of the proposition is to the following situation.

Corollary 3.10. *Let X be compact Hausdorff, and let A be a closed subspace. Then X/A is compact Hausdorff.*

Proof. All we need to do is show that the projection $p : X \to X/A$ is closed. Let E be a closed subset of X. Then

$$p^{-1}(p(E)) = \begin{cases} E & \text{if } E \cap A = \emptyset \\ E \cup A & \text{if } E \cap A \neq \emptyset. \end{cases}$$

In either case this set is closed, so $p(E)$ is closed. □

1.1. Projective Spaces. Let $X = \mathbf{R}^{n+1} - \{0\}$. The set of lines through the origin in $\mathbf{R}^{n+1}$ is called *real projective n space* and it is denoted $\mathbf{R}P^n$. (Algebraic geometers often denote it $\mathbf{P}^n(\mathbf{R})$.) It may be visualized as a quotient space as follows. Let $X = \mathbf{R}^{n+1} - \{0\}$, and consider points equivalent if they lie on the same line, i.e., one is a non-zero multiple of the other. Then clearly $\mathbf{R}P^n$ is the quotient space and as such is endowed with a topology. It is fairly easy to see that it is Hausdorff. (Any two lines in $\mathbf{R}^n - \{0\}$ can be chosen to be the axes of open double 'cones' which don't intersect.)

Here is another simpler description. (It is helpful to concentrate on $n = 2$, i.e., the real projective plane.) Consider the inclusion $i : S^n \to \mathbf{R}^{n+1} - \{0\}$ and follow this by the projection to $\mathbf{R}P^n$. This map is clearly onto. Since S^n is compact, and $\mathbf{R}P^n$ is Hausdorff, it is a quotient map by the proposition above. (It is also a closed map by the proof of the proposition. It is also open because the image of any open set U is the same as the image of $U \cup (-U)$ which is open.) Note also that distinct points in S^n are equivalent under the induced equivalence relation if and only if they are antipodal points $(x, -x.)$ Since S^n is compact, it follows that $\mathbf{R}P^n$ is compact and Hausdorff.

Here is an even simpler description. Let

$$X = \{x \in S^n \mid x_{n+1} \geq 0\}$$

(the upper hemisphere.) Repeat the same reasoning as above to obtain a quotient map of X onto $\mathbf{R}P^n$. Note that distinct points on the bottom edge (which we may identify as S^{n-1}) are equivalent if and only if they are antipodal. Points not on the edge are equivalent only to themselves, i.e., the quotient map is one-to-one for those points.

Finally, map D^n onto $\mathbf{R}P^n$ as follows. Imbed D^n in $\mathbf{R}^{n+1}$ in the usual way in the hyperplane $x_{n+1} = 0$. Project upward onto the upper hemisphere and then map onto $\mathbf{R}P^n$ as above. Again, this yields a quotient map which is one-to-one on interior points of D^n and such that antipodal points on the boundary S^{n-1} are equivalent.

There is an interesting way to visualize $\mathbf{R}P^2$. The unit square is homeomorphic to D^2, and if we identify the edges as indicated below, we get $\mathbf{R}P^2$.

We may now do a series of 'cuttings' and 'pastings' as indicated below. (A cutting exhibits a space as the quotient of another space which is a disjoint union of appropriate spaces.)

Note the use of the Moebius band described as a square with two opposite edges identified with reversed orientation. From this point of view, the real projective plane is obtained by taking a 2-sphere, cutting a hole, and pasting a Moebius band on the edge of the hole. Of course this can't be done in $\mathbf{R}^3$ since we would have to pass the Moebius band through itself in order to get its boundary (homeomorphic to S^1) lined up properly to paste onto the edge of the hole. A Moebius band inserted in a sphere in this way is often called a cross-cap.

1.2. Group Actions and Orbit Spaces. Let G be a group and X a set. A (left) group action of G on X is a binary operation $G \times X \to X$ (denoted here $(g, x) \mapsto gx$) such that

(i) $1x = x$ for every $x \in X$.
(ii) $(gh)x = g(hx)$ for $g, h \in G$ and $x \in X$. This is a kind of associativity law.

(There is a similar definition for a right action which I leave to your imagination.)

If G acts on X, then for each $g \in G$, there is a function $L(g) : X \to X$. The rules imply that $L(1) = \mathrm{Id}_X$ and $L(gh) = L(g) \circ L(h)$. Since $L(g) \circ L(g^{-1}) = L(1) = \mathrm{Id}_X$, it follows that each $L(g)$ is in fact a bijection. Hence, this defines a function $L : G \to \mathcal{S}(X)$, the group of all bijections of X with composition of functions the group0 operation. This function is in fact a homomorphism.

This formalism may in fact be reversed. Given a homomorphism $L : G \to S(X)$, we may define a group action of G on X by $gx = L(g)(x)$.

Let G act on X. The set $Gx = \{gx \mid g \in G\}$ is called the *orbit* of x. It is in fact an equivalence class of the following relation

$$x \sim y \Leftrightarrow \exists g \in G \quad \text{such that } y = gx.$$

(That this is an equivalence relation was probably proved for you in a previous course, but if you haven't ever seen it, you should check it now.)

Suppose now that X is a topological space and G acts on X. We shall require additionally that $L(g) : X \to X$ is a continuous map for each $g \in G$. As above, it is invertible and its inverse is continuous, so it is a homeomorphism. In this case, we get a homomorphism $L : G \to \mathcal{A}(X)$, the group of all homeomorphisms of X onto itself.

Form the quotient space $X/\sim$ for the equivalence relation associated with the group action. As mentioned above, it consists of the orbits of the action. It is usually denoted X/G (although there is a reasonable argument to denote it $G\backslash X$).

EXAMPLE 3.11. Let $G = \mathbf{Z}$ and $X = \mathbf{R}$. Let $\mathbf{Z}$ act on $\mathbf{R}$ by

$$n \cdot x = n + x.$$

This defines an action. It is a little confusing to check this because the group operation in $\mathbf{Z}$ is denoted additively, with the neutral element being denoted '0' rather than '1'.

(i) $0 \cdot x = x + 0 = x$.
(ii) $(n+m)\cdot x = x+n+m = (x+m)+n = n\cdot(x+m) = n\cdot(m\cdot x)$.

The orbit of a point x is the set of all integral translates of that point. The quotient space is homeomorphic to S^1. This is easy to see by noting that the exponential map $E \to S^1$ defined earlier is in fact a quotient map with the sets $E^{-1}(z), z \in S^1$ being the orbits of this group action.

EXAMPLE 3.12. We can get a similar action by letting $\mathbf{Z}^n$ act on $\mathbf{R}^n$ by $\mathbf{n} \cdot \mathbf{x} = \mathbf{x} + \mathbf{n}$. The quotient space is the n-torus $(S^1 i)^n$.

EXAMPLE 3.13. Examples 3.11 and 3.12 are special cases of the following general construction. Assume X is a group in which the group operation is a continuous function $X \times X \to X$. Let G be a subgroup in the ordinary sense. Define an action by letting gx be the ordinary composition in X. The the orbits are the *right* cosets Gx of G in X, and the orbit space is the set of such cosets. If G is a normal subgroup, as would always be the case if the group were abelian, then X/G is just the quotient group.

EXAMPLE 3.14. Let X be any space, and consider the n-fold cartesian product $X^n = X \times X \times \cdots \times X$. Consider the symmetric group $\mathcal{S}_n$ of all permutations of $\{1, 2, \ldots, n\}$. Define an action of $\mathcal{S}_n$ on X^n by

$$\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}$$

for $\sigma \in \mathcal{S}_n$. Note the use of σ^{-1}. You should check the associativity rule here! The resulting orbit space is called a symmetric space. The case $n = 2$ is a bit easier to understand since $\sigma^{-1} = \sigma$ for the one non-trivial element of $\mathcal{S}_2$.

EXAMPLE 3.15. Let $\mathcal{S}_2 = \{1, \sigma\}$ act on S^n by $\sigma(x) = -x$. Then the orbits are pairs of antipodal points, and the quotient space is $\mathbf{R}P^n$. (For $n = 1$, the space S^1 has a group structure (multiplication of complex numbers of absolute value 1), and we can identify σ with the element -1, so $\mathcal{S}_2$ may be viewed as a subgroup of S^1. Strangely enough, this also works for S^3, but you have to know something about quaternions to understand that.)

EXAMPLE 3.16. Let $z_n = E(1/n) = e^{2\pi i/n} \in S^1$. The subgroup C_n of S^1 generated by z_n is cyclic of order n. It is not hard to see that the orbit space S^1/C_n is homeomorphic to S^1 again. Indeed a quotient map $p_n : S^1 \to S^1$ is given by $p_n(z) = z^n$. Each orbit consists of n points.

2. Covering Spaces

Let X be a path connected space. A map $p : \tilde{X} \to X$ from a path connected space $\tilde{X}$ is called a *covering map* (with $\tilde{X}$ being called a covering space) if for each point $x \in X$, there is an open neighborhood U of x such that

$$p^{-1}(U) = \bigcup_i S_i$$

is a disjoint union of open subsets of $\tilde{X}$, and for each i, $p|S_i$ is a homeomorphism of S_i onto U. An open neighborhood with this property is called *admissible*. Note that the set $p^{-1}(x)$ (called the fiber at x) is necessarily a discrete subspace of $\tilde{X}$.

EXAMPLE 3.17. Let $X = S^1, \tilde{X} = \mathbf{R}$, and $p = E$ the exponential map. More generally, let $X = (S^1)^n$ be an n-torus and let $\tilde{X} = \mathbf{R}^n$. The map E^n is a covering map.

EXAMPLE 3.18. Let $X = S^1$ (imbedded in $\mathbf{C}$), and let $\tilde{X}$ also be S^1. Let $p(z) = z^n$. This provides an n-fold covering of S^1 by itself.

EXAMPLE 3.19. Let $X = \mathbf{C}, \tilde{X} = \mathbf{C}$ and define $p : \mathbf{C} \to \mathbf{C}$ by $p(z) = z^n$. This is not a covering map. Can you see why? What happens if you delete $\{0\}$?

EXAMPLE 3.20. Let $\tilde{X} = S^n$ and $X = \mathbf{R}P^n$. Let p be the quotient map discussed earlier. This provides a two sheeted covering. For, if y is any point on S^n, we can choose an open neighborhood U_y which is disjoint from $-U_y$. Then $p(U_y)$ is an open neighborhood of $p(y)$, and $p^{-1}(p(U_y)) = U_y \cup (-U_y)$.

Let Z be a connected space, $z_0 \in Z$, and let $f : (Z, z_0) \to (X, x_0)$. A map $\tilde{f} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ is said to lift f if $f = p \circ \tilde{f}$.

PROPOSITION 3.21 (Uniqueness of liftings). *Let $\tilde{f}, \tilde{g} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ both lift f. Then $\tilde{f} = \tilde{g}$.*

.

PROOF. First, we show that the set $W = \{z \in Z \mid \tilde{f}(z) = \tilde{g}(z)\}$ is open. Let $z \in W$. Choose an admissible open neighborhood U of $f(z) \in X$, so $p^{-1}(U) = \cup_i S_i$ as above. Suppose $\tilde{f}(z) = \tilde{g}(z) \in S_i$. Let $V = \tilde{f}^{-1}(S_i) \cup \tilde{g}^{-1}(S_i)$. V is certainly an open set in Z. Moreover, for any point $z' \in V$, we have $p(\tilde{f}(z')) = f(z') = p(\tilde{g}(z'))$. Since $\tilde{f}(z'), \tilde{g}(z') \in S_i$, and p is one-to-one on S_i, it follows that $\tilde{f}(z') = \tilde{g}(z')$. Hence, $V \subseteq W$. This shows W is open.

W is certainly non-empty since it contains z_0. Hence, if we can show its complement $W' = \{z \in Z \mid \tilde{f}(z) \neq \tilde{g}(z)\}$ is also open, we can conclude it must be empty by connectedness. But, it is clear that W' is open. For, if $z \in W'$, $\tilde{f}(z)$ and $\tilde{g}(z)$ must be in disjoint components S_i and S_j of $p^{-1}(f(z)$. But then, the same is true for every point in $\tilde{f}^{-1}(S_i) \cup \tilde{g}^{-1}(S_j)$.

$\square$

PROPOSITION 3.22 (Lifting of paths). *Let $p : \tilde{X} \to X$ be a covering map. Let $h : I \to X$ be a path starting at x_0. Let $\tilde{x}_0$ be a point in $\tilde{X}$ over x_0 Then there is a unique lifting $\tilde{h} : (I, 0) \to (\tilde{X}, \tilde{x}_0)$, i.e., such that $h = p \circ \tilde{h}$ and $\tilde{h}$ starts at $\tilde{x}_0$.*

PROOF. The uniqueness has been dealt with.

For each $x \in X$ choose an admissible open neighborhood U_x of x. Apply the Lebesgue Covering Lemma to the covering $I = \cup_t h^{-1}(U_x)$. It follows that there is a partition

$$0 = t_0 < t_1 < t_2 \cdots < t_n = 1$$

such that for each $i = 1, \ldots, n$, $[t_{i-1}, t_i] \subseteq h^{-1}(U_i)$ for some admissible open set U_i in X, i.e., $h([t_{i-1}, t_i]) \subseteq U_i$. Let h_i denote the restriction of h to $[t_{i-1}, t_i]$, Choose the component S_1 of $p^{-1}(U_1)$ containing $\tilde{x}_0$. (Recall that $p(\tilde{x}_0) = x_0 = h(0)$.) Let $\tilde{h}_1 = p_1^{-1} \circ h_1$ where $p_1 : S_1 \to U_1$ is the restriction of the covering map p. Let $x_1 = h(t_1)$ and $\tilde{x}_1 = \tilde{h}_1(t_1)$. Repeat the argument for this configuration. We get a lifting $\tilde{h}_2$ of h_2 such that $\tilde{h}_1(t_1) = \tilde{h}_2(t_1)$. Continuing in this way, we get a lifting $\tilde{h}_i$ for each h_i, and these liftings agree at the endpoints of the intervals. Putting them together yields a lifting $\tilde{h}$ for h such that $\tilde{h}(0) = \tilde{x}_0$. □

PROPOSITION 3.23 (Homotopy Lifting Lemma). *Let $p : \tilde{X} \to X$ be a covering map. Suppose $F : Z \times I \to X$ is a map such that $f = F(-, 0) : Z \to X$ can be lifted to $\tilde{X}$, i.e., there exists $\tilde{f} : Z \to \tilde{X}$ such that $f = p \circ \tilde{f}$. Then F can be lifted consistently, i.e., there exists $\tilde{F} : Z \times I \to \tilde{X}$ such that $F = p \circ \tilde{F}$ and $\tilde{f} = \tilde{F}(-, 0)$.*

Moreover, if Z is connected then $\tilde{F}$ is unique.

PROOF. The uniqueness in the connected case follows from the general uniqueness proposition proved above.

To show existence, consider first the case in which $F(Z \times I) \subseteq U$ where U is an admissible open set in X. Let $p^{-1}(U) = \cup_i S_i$ as usual, and let $p_i = p|S_i$. The sets $Z_i = \tilde{f}^{-1}(S_i)$ provide a covering of Z by *disjoint* open sets. $(f(Z_i \cup Z_j) \subseteq S_i \cup S_j = \emptyset.)$ Hence, if we define

$$\tilde{F}(z, t) = p_i{}^1(F(z, t) \qquad \text{for } z \in V_i$$

we will never get a contradiction, and clearly $\tilde{F}$ lifts F.

Also, for $z \in V_i$ (unique for z), we have $p_i(\tilde{F}(z, 0)) = F(z, 0) = f(z)$, so since p_i is one-to-one, we have $\tilde{F}(z, 0) = \tilde{f}(z)$. (Note that this argument would be much simpler if Z were connected.)

Consider next the general case. For each $z \in Z, t \in I$, choose an admissible neighborhood $U_{z,t}$ of $F(z, t)$. Choose an open neighborhood

$Z_{z,t}$ of z and a closed interval $I_{z,t}$ containing t such that $F(Z_{z,t} \times I_{z,t}) \subseteq U_{z,t}$.

Fix one z. The sets $I_{z,t}$ cover I, so we may pick out a finite subset of them $I_1 = I_{t_1}, I_2 = I_{t_2}, \dots, I_k = I_{t_k}$ which cover I. Let $V = \cup_j Z_{z,t_j}$. By the Lebesgue Covering Lemma applied to I, we can find a partition $0 = s_0, s_1 < \cdots < s_n = 1$ such that each $J_i = [s_{i-1}, s_i]$ is contained in some I_j. Then, each $F(V \times J_i)$ is contained in an admissible neighborhood of X, so we may apply the previous argument (with J_i replacing I). First lift $F|V \times J_1$ to $\tilde{F}_1$ so that $\tilde{F}_1(v, 0) = \tilde{f}(v)$ for $v \in V$. Next lift $F|V \times J_2$ so that $\tilde{F}_2(v, s_1) = \tilde{F}_1(v, s_1)$ for $v \in V$. Continue in this way until we have liftings $\tilde{F}_i$ for each i. Gluing these together we get a lifting $\tilde{F}_V : V \times I \to \tilde{X}$ which agrees with $\tilde{f}|V$ for $s = 0$.

We shall now show that these $\tilde{F}_V$ are consistent with one another on intersections. (So they define a map $\tilde{F} : Z \times I \to \tilde{X}$ with the right properties by the gluing lemma.) Let $v \in V \cup W$ where V and W are appropriate open sets in Z as above. Consider $\tilde{F}_V(v, -) : I \to \tilde{X}$ and $\tilde{F}_W(v, -: I \to \tilde{X}$. These both cover $F(v, -) : I \to X$ and for $s = 0$, $\tilde{F}_V(v, 0) = \tilde{f}(v) = \tilde{F}_W(v, 0)$. Since I is connected, the uniqueness proposition implies that $\tilde{F}_V(v, s) = \tilde{F}_W(v, s)$ for all $s \in I$. However, since v was an arbitrary element of $V \cup W$, we are done. $\square$

PROPOSITION 3.24. *Let $p : \tilde{X} \to X$ be a covering map. Let $H : I \times I \to X$ be a homotopy relative to $\dot{I}$ of paths $h, h' : I \to X$ which start and end at the same points x_0 and x_1. Let $\tilde{h}$ and $\tilde{h}'$ be liftings of h and h' respectively which start at the same point $\tilde{x}_0 \in \tilde{X}$ over x_0. Then there is a lifting $\tilde{H} : I \times I \to \tilde{X}$ which is a homotopy relative to $\dot{I}$ of $\tilde{h}$ to $\tilde{h}'$. In particular, $\tilde{h}(1) = \tilde{h}'(1)$.*

PROOF. This mimics the proof in the case of the covering $\mathbf{R} \to S^1$ which we did previously. Go back and look at it again.

Since $\tilde{H}(0,s)$ lies over $h(0) = h'(0)$ for each s, the image of $\tilde{H}(0,-)$ is contained in a discrete space (the fiber over x_0) so it is constant. A similar argument works for $\tilde{H}(1,-)$. By construction, $\tilde{H}(-,0) = \tilde{h}$. Similarly, $\tilde{H}(-,1)$ and $\tilde{h}'$ both lift $H(-,1) = h'$ and they both start at $\tilde{x}_0$, so they are the same. □

COROLLARY 3.25. *Let $p : \tilde{X} \to X$ be a covering map, and choose $\tilde{x}_0$ over x_0. Then $p_* : \pi(\tilde{X}, \tilde{x}_0) \to \pi(X, x_0)$ is a monomorphism, i.e., a one-to-one homomorphism.*

PROOF. based at $\tilde{x}_0$. If $p \circ \tilde{h}$ and $p \circ h'$ are homotopic in X relative to $\dot{I}$, then by the lifting homotopy lemma, so are $\tilde{h}$ and $\tilde{h}'$. □

3. Action of the Fundamental Group on Covering Spaces

Let $p : \tilde{X} \to X$ be a covering map, fix a point $x \in X$ and consider $\pi(X, x)$. We can define a *right* action of $\pi(X, x)$ on the fiber $p^{-1}(x)$ as follows. Let $\alpha \in \pi(X, x_0)$ and let $\tilde{x}$ be a point in $\tilde{X}$ over x. Let $h : I \to X$ be a loop at x which represents α. By the lifting lemma, we may lift h to a path $\tilde{h} : I \to \tilde{X}$ starting at $\tilde{x}$ and which by the homotopy lifting lemma is unique up to homotopy relative to $\dot{I}$. In particular, the endpoint $\tilde{h}(1)$ depends only on α and $\tilde{x}$. Define

$$\tilde{x}\alpha = \tilde{h}(1).$$

This defines a right $\pi(X, x)$ action. For, the trivial element is represented by the trivial loop which lifts to the trivial loop at $\tilde{x}$; hence, the trivial element acts trivially. Also, if β is another element of $\pi(X, x)$, represented say by a loop g, then we may lift $h * g$ by first lifting h to $\tilde{h}$ starting at $\tilde{x}$ and then lifting g to $\tilde{g}$ starting at $\tilde{h}(1)$. It follows that

$$\tilde{x}(\alpha\beta) = (\tilde{x}\alpha)\beta$$

as required for a right action.

We shall see later that this action can be extended to an action of the fundamental group on $\tilde{X}$ provided we make plausible further assumptions about $\tilde{X}$ and X.

Before proceeding, we need some more of the machinery of group actions. Before, we discussed generalities in terms of left actions, so for variation we discuss further generalities using the notation appropriate for right actions. (But, of course, with obvious notational changes it doesn't matter which side the group acts on.) So, let G act on X on the right. In this case an orbit would be denoted xG. We say the action on the set is transitive if there is only one orbit. Another way to say that is

$$\text{for each } x, y \in X, \qquad \exists g \in G \qquad \text{such that } y = xg.$$

(For example, the symmetric group $\mathcal{S}_3$ certainly acts transitively on the set $\{1, 2, 3\}$ but so does the cyclic group of order 3 generated by the cycle (123). On the other hand, the cyclic subgroup generated by the transposition (12) does not act transitively. In the latter case, the orbits are $\{1, 2\}$ and $\{3\}$.)

Proposition 3.26. *Let $p : \tilde{X} \to X$ be a covering map, and let $x \in X$. The action of $\pi(X, x)$ on $p^{-1}(x)$ is transitive.*

Proof. Let $\tilde{x}$ and $\tilde{y}$ lie over x. By assumption, since p is a covering, $\tilde{X}$ is path connected, so there is a path $\tilde{h} : I \to \tilde{X}$ starting at $\tilde{x}$ and ending at $\tilde{y}$. The projected path $p \circ \tilde{h}$ is a loop based at x, so it represents some element $\alpha \in \pi(X, x)$. From the above definition, $\tilde{y} = \tilde{x}\alpha$. □

Continuing with generalities, let G act on a set X. If $x \in X$, consider the set $G_x = \{g \in G \mid xg = x\}$. We leave it to the student to check that G_x is a subgroup of G. It is called the isotropy subgroup of the point x. (It is also sometimes called the stabilizer of the point x.) Let G/G_x denote the set of *right* cosets of G_x in G. (In the case of a left action, we would consider the set of left cosets instead.) Note that G/G_x isn't generally a group (unless G_x happens to be normal), but that we do have a right action of G on G/G_x. Namely, if H is any subgroup of G, the formula

$$(Hc)g = H(cg)$$

defines a right action of G on G/H. (There are some things here to be checked. First, you must know that the quantity on the right depends only on the *coset* of c, not on c. Secondly, you must check that the formula does define an action. We leave this for you to verify, but you may very well have seen it in an algebra course.)

Proposition 3.27. *Let G act on X (on the right), and let $x \in X$. Then $G_xc \mapsto xc$ defines an injection $\phi : G/G_x \to X$ with image the orbit xG. Moreover, ϕ is a map of G-sets, i.e., $\phi(\bar{c}g) = \phi(\bar{c})g$ for $\bar{c} = G_xc$ and $g \in G$.*

It follows that the index $(G : G_x)$ equals the cardinality of the orbit $|xG|$.

Note that the index and cardinality of the orbit could be transfinite cardinals, but of course to work with that you would have to be familiar with the theory of infinite cardinals. For us, the most useful case is that in which both are finite.

PROOF. First note that ϕ is well defined. For, suppose $G_x c = G_x d$. Then $cd^{-1} \in G_x$, i.e., $x(cd^1) = x$. From this is follows that $xc = xd$ as required. It is clearly onto the orbit xG. To see it is one-to-one, suppose $xc = xd$. Then $x(cd^{-1} = x$, whence $cd^{-1} \in G_x$, so $G_x c = G_x d$. We leave it as an exercise for the student to check that ϕ is a map of G sets. □

Suppose x, y are in the same orbit with $y = xh$. Then

$$g \in G_y \Leftrightarrow (xh)g = xh \Leftrightarrow x(hgh^{-1}) = x \Leftrightarrow hgh^{-1} \in G_x.$$

This shows that

PROPOSITION 3.28. *Isotropy subgroups of points in the same orbit are conjugate, i.e.,* $G_{xh} = h^{-1} G_x h$.

We can make this a bit more explicit in the case of $\pi(X, x)$ acting on $p^{-1}(x)$.

COROLLARY 3.29. *Let* $p : \tilde{X} \to X$ *be a covering. The isotropy subgroup of* $\tilde{x} \in p^{-1}(x)$ *in* $\pi(X, x)$ *is* $p_*(\pi(\tilde{X}, \tilde{x}))$. *In particular,*

$$(\pi(X, x) : p_*(\pi(\tilde{X}, \tilde{x})) = |p^{-1}(x)|.$$

Moreover, if $\tilde{y}$ *is another point in* $p^{-1}(x)$ *then*

$$p_*(\pi(\tilde{X}, \tilde{y})) = \beta^{-1} p_*(\pi(\tilde{X}, \tilde{x}))\beta$$

where $\beta \in \pi(X, x)$ *is represented by the projection of a path in* $\tilde{X}$ *from* $\tilde{x}$ *to* $\tilde{y}$.

PROOF. This is just translation. Note that the β in the second part of the Corollary is chosen so that $\tilde{y} = \tilde{x}\beta$. □

As mentioned above, in the proper circumstances, the action of the fundamental group on fibers is part of an action on the covering space. Even without going that far, we can show that actions on different fibers are essentially the same. To see this, let $x, y \in X$ be two different points. Let $h : I \to X$ denote a path in X from x to y. Then, we considered before the isomorphism $\phi_h : \pi(X, x) \to \pi(X, y)$. Choose a path $\tilde{h} : I \to \tilde{X}$ over h and suppose it starts at $\tilde{x}$ over x and ends at $\tilde{y}$ over y.

PROPOSITION 3.30. *With the above notation, the following diagram commutes*

$$\begin{tikzcd} \pi(\tilde{X},\tilde{x}) \arrow[r, "\phi_{\tilde{h}}"] \arrow[d, "p_*"'] & \pi(\tilde{X},\tilde{y}) \arrow[d, "p_*"] \\ \pi(X,x) \arrow[r, "\phi_h"] & \pi(X,y) \end{tikzcd}$$

PROOF. Let $\tilde{g}$ be a loop at $\tilde{x}$. Then

$$p \circ (\tilde{h} * \tilde{g}\tilde{h}') = (p \circ \tilde{h}) * (p \circ \tilde{g}) * (p \circ \tilde{h}') = h \circ (p \circ \tilde{g}) \circ h'$$

(where h' as before denotes the reverse path to h.) This says on the level of paths exactly what we want. $\square$

COROLLARY 3.31. *If $p : \tilde{X} \to X$ is a covering, then the fibers at any two points have the same cardinality.*

PROOF. By the proposition, the isomorphism $\phi_h : \pi(X,x) \to \pi(X,y)$ carries $p_*\pi(\tilde{X},\tilde{x})$ onto $p_*\pi(\tilde{X},\tilde{y})$. Hence,

$$|p^{-1}(x)| = (\pi(X,x) : p_*(\pi(\tilde{X},\tilde{x})) = (\pi(X,y) : p_*(\pi(\tilde{X},\tilde{y})) = |p^{-1}(y)|.$$

$\square$

The common number $|p^{-1}(x)|$ is called the number of sheets of the covering. For example the map $p_n : S^1 \to S^1$ defined by $p_n(z) = z^n$ provides an n-sheeted covering. Similarly, $S^1 \to \mathbf{R}P^n$ is a 2-sheeted covering for any $n \geq 1$.

EXAMPLE 3.32. The above analysis shows that

$$\pi(\mathbf{R}P^n, x_0) \cong \mathbf{Z}/2\mathbf{Z} \qquad n \geq 2$$

for any base point x_0. The argument is that

$$(\pi(\mathbf{R}P^n, x_0) : p_*(\pi(S^n, \tilde{x}_0)) = |p^{-1}(x_0)| = 2.$$

However, since S^n is simply connected, $\pi(S^n, \tilde{x}_0) = \{1\}$, so $\pi(\mathbf{R}P^n, x_0)$ has order 2.

A covering $p : \tilde{X} \to X$ is called a universal covering ($\tilde{X}$ a universal covering space) if $\tilde{X}$ is simply connected. Note that in this case

$$|p^{-1}(x)| = |\pi(X,x)|.$$

EXAMPLE 3.33. $E^n : \mathbf{R}^n \to T^n = (S^1)^n$ is a universal covering. So is $S^n \to \mathbf{R}P^n$. However, $S^1 \to S^1$ defined by $z \mapsto z^n$ is not.

4. Existence of Coverings and the Covering Group

Let X be path connected and fix $x_0 \in X$. The collection of covering maps $p : \tilde{X} \to X$ form a category. The objects are the covering maps. Given two such maps $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$, a morphism from p to p' is a map $f : \tilde{X} \to \tilde{X}'$ such that $p = p' \circ f$, i.e.,

commutes. Such morphisms are called 'maps over X'. Similarly, we can consider the category of coverings with basepoint $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, where the morphisms are maps over X preserving basepoints.

In the basepoint preserving category, the uniqueness lemma assures us that if there is a map $f : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}', \tilde{x}'_0)$ over X, it is unique. In this case, we may say the first covering with base point *dominates* the other. Domination behaves like a partial order on the collection of coverings in the sense that it is reflexive and transitive—which you should prove—but two coverings can dominate each other without being the same. On the other hand, in the first category (ignoring base points), there may be many maps between objects. In particular, we may consider the collection of *homeomorphisms* $f : \tilde{X} \to \tilde{X}$ over X. This set forms a group under composition. For it is clear that it is closed under composition, that $\mathrm{Id}_{\tilde{X}}$ is in it, and that the inververse of a map over X is a map over X. This group is called the covering group of the covering and denoted $\mathrm{Cov}_X(\tilde{X})$.

Proposition 3.34. *Let $p : \tilde{X} \to X$ be a covering map, and let $x \in X$. The actions of $\mathrm{Cov}_X(\tilde{X})$ and $\pi(X, x)$ on the fiber $p^{-1}(x)$ are consistent, i.e.,*

$$f(\tilde{x}\alpha) = f(\tilde{x})\alpha$$

for $f \in \mathrm{Cov}_X(\tilde{X}), \alpha \in \pi(X, x), \tilde{x} \in p^{-1}(x)$.

Proof. Let $[h] = \alpha$ where h is a loop in X at x. Let $\tilde{h}$ be the unique path over h starting at $\tilde{x}$. Since $p = p \circ f$, $f \circ \tilde{h}$ is the unique path over h starting at $f(\tilde{x})$. We have

$$f(\tilde{x})\alpha = (f \circ \tilde{h})(1) = f(\tilde{h}(1)) = f(\tilde{x}\alpha).$$

$\square$

In this rest of this section we want to explore further the relation between these actions. We shall see that in certain circumstances, they are basically the same.

The first question we shall investigate is the existence of maps between covering spaces. We state the relevant lemma in somewhat broader generality.

PROPOSITION 3.35 (Existence of liftings). *Let $p : \tilde{X} \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0$ lie over x_0. Let $f : (Z, z_0) \to (X, x_0)$ be a basepoint preserving map of a connected, locally path connected space Z into X. Then f can be lifted to a map $\tilde{f} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if*

$$f_*(\pi(Z, z_0)) \subseteq p_*(\pi(\tilde{X}, x_0).$$

Recall: A space Z is locally path connected if given any point $z \in Z$ and a neighborhood V of z, there is a smaller open neighborhood $W \subseteq V$ of z which is path connected. A space can be path connected without being locally path connected. (Look at the homework problems. A space introduced in another context is an example. Which one is it?) However, if Z is locally path connected, then it is connected if and only if it is path connected.

The most important application of the above Proposition is to the case in which Z is simply connected because then the condition is certainly verified. In particular, suppose X is locally path connected, so in fact any covering space is locally path connected. Suppose in addition that there is a universal covering $p : \tilde{X} \to X$. According to this proposition, this maps over $\tilde{X}$ to any other covering, and in the base point preserving category, such a map is unique. Hence, under these hypotheses, a univeral covering space with base point is a 'largest' object in the sense that it dominates every other object.

PROOF. If such a map $\tilde{f}$ exists, it follows from $f_* = p_* \circ \tilde{f}_*$ that the required relation holds between the two images in $\pi(X, x_0)$.

Define $\tilde{f} : (Z, z_0) \to (\tilde{X}, x_0)$ as follows. Let $z \in Z$. Choose a path h in Z from z_0 to z. (Z is path connected.) Lift $f \circ h$ to a path $\tilde{g} : I \to \tilde{X}$ such that $\tilde{g}(0) = \tilde{x}_0$, and let

$$\tilde{f}(z) = \tilde{g}(1).$$

If this function is well defined and continuous, it satisfies the desired conctions. For, if $z = z_0$, we may choose the trivial path at z_0, so $\tilde{f}(z_0) = \tilde{x}_0$. Also,

$$p(\tilde{f}(a)) = p(\tilde{g}(1)) = f(h(1)) = f(z)$$

so $\tilde{f}$ covers f.

It remains to show that the above definition is that of a continuous map. First we note that it is well defined. For, suppose $h' : I \to Z$ is another path from z_0 to z. Then $h' * \overline{h}$ is a loop in Z at z_0.

Thus,

$$(f \circ h') * (\overline{f \circ h}) = f \circ (h' * \overline{h}) \sim_{\dot{I}} p \circ \tilde{j}$$

for some loop $j : I \to \tilde{X}$ at $\tilde{x}_0$. (That is a translation of the statement that $\operatorname{Im} f_* \subseteq \operatorname{Im} p_*$.). Hence,

$$f \circ h' \sim_{\dot{I}} (p \circ \tilde{j}) * (f \circ h) = (p \circ \tilde{j}) * (p \circ \tilde{g}) = p \circ (\tilde{j} * \tilde{g}).$$

Hence, by the homotopy lifting lemma, any lifting $\tilde{g}'$ of $f \circ h'$ starting at $\tilde{x}_0$ must end in the same place as $\tilde{g}$, i.e., $\tilde{g}'(1) = \tilde{g}(1)$. Thus the function $\tilde{f}$ is well defined. To prove $\tilde{f}$ is continuous, we need to use the hypothesis that Z is locally path connected. Let $\tilde{U}$ be an open set in $\tilde{X}$. We shall show that $\tilde{f}^{-1}(\tilde{U})$ is open. Let $z \in \tilde{f}^{-1}(\tilde{U})$. Choose an admissible open neighborhood W of $f(z)$ and let S be the component of $p^{-1}(W)$ containing $\tilde{f}(z)$. Then since $p|S$ is a homemorphism, the set $p(S \cap \tilde{U})$ is and open neighborhood of $f(z)$. Hence $f^{-1}(p(S \cap \tilde{U}))$ is an open neighborhood of z and we may choose a path connected open neighborhood V of z contained in it. Let $v \in V$. Choose a path h from z_0 to z, a path k in V from z to v and let $h' = h * k$.

Lift $f \circ h'$ as follows. First lift $f \circ h$ to $\tilde{g}$ as before so $\tilde{f}(z)$ is the endpoint of $\tilde{g}$. Inside $S \cap \tilde{U}$ lift $f \circ k$ which is a path in $p(S \cap \tilde{U})$ by composing with $(p|S)^{-1}$ (which is a homeomorphism). The result $\tilde{l}$ will be a path in $S \cap \tilde{U}$ starting at the endpoint of $\tilde{g}$. Hence, the path $\tilde{g} * \tilde{l}$ lifts $f \circ (h * k) = (f \circ h) * (f \circ k)$. It follows that $\tilde{f}(v)$ which is the endpoint of $\tilde{g} * \tilde{l}$ lies in $S \cap \tilde{U}$. Hence, $V \subseteq \tilde{f}^{-1}(\tilde{U})$, which shows that every point in $\tilde{f}^{-1}(\tilde{U})$ has an open neighborhood also contained in that set. Hence, $\tilde{f}^{-1}(\tilde{U})$ is open as required. $\square$

The proposition gives us a better way to understand the category of coverings with base point $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ of X. There is a (unique) map in this category $f : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}'_0)$, i.e., the first covering dominates the second, if and only if $p_*(\pi(\tilde{X}, \tilde{x}_0)) \subseteq p'_*(\pi(\tilde{Y}, \tilde{y}_0))$. If the images of the two fundamental groups are equal, each dominates the other, which is to say that they are isomorphic objects in the category of coverings with base points. Thus, there is a one-to-one correspondence between isomorphism classes of coverings with basepoints and a certain collection of subgroups of $\pi(X, x_o)$. (We shall see later that if there is a universal covering space, then every subgroup arises in this way.) Moreover, the ordering of such isomorphism classes under domination is reflected in the ordering of subgroups under inclusion.

The category of coverings (ignoring basepoints) is a bit more complicated. If $p : \tilde{X} \to X$ is a covering, then by a previous propostion (—find it—) the subgroups $p_*(\pi(\tilde{X}, \tilde{x}_0))$ for $\tilde{x}_0 \in p^{-1}(x_0)$ form a complete set of conjugate subgroups of $\pi(X, x_0)$, i.e., a *conjugacy class*. If $p : \tilde{X}' \to X$ is another covering which yields the same conjugacy class, then each $p_*(\pi(\tilde{X}', \tilde{x}'_0))$ (for $\tilde{x}'_0 \in p'^{-1}(x_0)$) is conjugate to some $p_*(\pi(\tilde{X}, \tilde{x}_0))$, i.e.,

$$p_*(\pi(X', \tilde{x}'_0)) = \beta^{-1} p_*(\pi(\tilde{X}, \tilde{x}_0))\beta = p_*(\pi(\tilde{X}, \tilde{x}_0\beta)).$$

It follows from the existence lemma above that there is an isomorphism $f : \tilde{X}' \to \tilde{X}$ over X (carrying $\tilde{x}'_0$ to $\tilde{x}_0\beta$). Thus, the isomorphism classes of coverings over X are in one-to-one correspondence with a certain collection of conjugacy classes of subgroups of $\pi(X, x_0)$. (Again, we shall see later that every such conjugacy class arises from a covering if X has a universal covering space.)

Return now to a single covering $p : \tilde{X} \to X$, and consider the covering group $\mathrm{Cov}_X(\tilde{X})$. Fix a base point $x \in X$ and $\tilde{x} \in \tilde{X}$ over x. Let $f \in \mathrm{Cov}_X(\tilde{X})$. Then by the uniqueness lemma, f is completely determined by the image $f(x)$. Also,

$$f(\tilde{x}) = \tilde{x}\gamma$$

for some $\gamma \in \Pi = \pi(X, x)$. Of course, γ is not unique. However,

$$\tilde{x}\gamma = \tilde{x}\delta \Leftrightarrow \tilde{x} = \tilde{x}\gamma\delta^{-1} \Leftrightarrow \gamma\delta^{-1} \in \Pi_{\tilde{x}} = p_*(\pi(\tilde{X}, \tilde{x})).$$

Hence, each f uniquely determines a coset $(\Pi_{\tilde{x}})\gamma$. Clearly, not every coset in $\Pi/\Pi_{\tilde{x}}$ need arise in this way. Indeed, there is a map $f : \tilde{X} \to \tilde{X}$ over X carrying $\tilde{x}$ to $\tilde{x}\gamma$ if and only if $p_*(\pi(\tilde{X}, \tilde{x})) = \Pi_{\tilde{x}} \subseteq p_*(\pi(\tilde{X}, \tilde{x}\gamma)) = \Pi_{\tilde{x}\gamma} = \gamma^{-1}\Pi_{\tilde{x}}\gamma$. However, this is the same as saying

$$\gamma\Pi_{\tilde{x}}\gamma^{-1} \subseteq \Pi_{\tilde{x}}.$$

The set of all γ with this property is called the normalizer of $\Pi_{\tilde{x}}$ in Π and is denoted $N_{\Pi}(\Pi_{\tilde{x}})$. It is easy to check that it is a subgroup of Π. In fact, it is the largest subgroup of Π which $\Pi_{\tilde{x}}$ is normal in. We have now almost proved the following proposition.

PROPOSITION 3.36. *Let X be a locally path connected, connected space, and let $p : \tilde{X} \to X$ be a covering. Let $x \in X$ and let $p(\tilde{x}) = x$. Then*

$$\mathrm{Cov}_X(\tilde{X}) \cong N_{\pi(X,x)}(p_*(\pi(\tilde{X}, \tilde{x})))/p_*(\pi(\tilde{X}, \tilde{x})).$$

PROOF. We need only prove that the map $f \mapsto (\Pi_{\tilde{x}})\gamma$ defined by $f(\tilde{x}) = \tilde{x}\gamma$ is a homomorphism. Let f' be another element of $\mathrm{Cov}_X(\tilde{X})$. Then

$$f'(f(\tilde{x})) = f'(\tilde{x}\gamma) = f'(\tilde{x})\gamma = (\tilde{x}\gamma')\gamma = \tilde{x}(\gamma'\gamma).$$

Hence $f' \circ f \mapsto (\Pi_{\tilde{x}})\gamma'\gamma$ as required. □

Note that much of this discussion simplifies in case $\pi(X, x)$ is abelian. For, in that case every subgroup is normal, and a conjugacy class consists of a single subgroup. Each covering corresponds to a single subgroup of the fundamental group and the covering group is the quotient group.

4.1. Existence of Covering Spaces. We now address the question of how to construct covering spaces in the first place. One way to do this is to start with a covering $p : \tilde{X} \to X$ and to try to construct coverings it dominates (after choice of a base point.) Suppose in particular that the covering has the property that $\Pi_{\tilde{x}} = p_*(\pi(\tilde{X}, \tilde{x}))$ is a *normal* subgroup of $\Pi = \pi(X, x)$. We call such a covering a *regular* or *normal* covering. Note in particular that any univeral covering is necessarily normal.

PROPOSITION 3.37. *Assume X is connected and locally path connected. Let $p : \tilde{X} \to X$ be a normal (regular) covering, and let $p(\tilde{x}) = x$. Then*

$$\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)/p_*(\pi(\tilde{X}, \tilde{x})).$$

For a universal covering, we have

$$\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x).$$

EXAMPLE 3.38. $E^n : \mathbf{R}^n \to T^n$ is a normal covering, so

$$\mathrm{Cov}_{T^n}(\mathbf{R}^n, \tilde{x}) \cong \pi(T^n, x) \cong \mathbf{Z}^n.$$

In this case the action is fairly easy to describe. Take $\mathbf{j} = (j_1, j_2, \dots, j_n) \in \mathbf{Z}^n$. Then $\mathbf{j} \cdot \mathbf{x} = \mathbf{j} + \mathbf{x}$. (If you look back at the proof that $\pi(S^1) \cong \mathbf{Z}$, which was done by liftings, you will see that we verified this in essence for $n = 1$. You should check that this reasoning can be carried through for $n > 1$.)

$p : S^n \to \mathbf{R}P^n$ for $n > 1$ is a universal covering, so $\mathrm{Cov}_{\mathbf{R}P^n}(S^n) \cong \mathbf{Z}/2\mathbf{Z}$. It is clear that the antipodal map is a nontrivial element of the covering group, so the covering group consists of the identity and the antipodal map.

PROPOSITION 3.39. *Suppose $p : \tilde{X} \to X$ is a covering where X is locally path connected and connected. Then the action of $G = \mathrm{Cov}_X(\tilde{X})$ on $\tilde{X}$ has the following property: for any $\tilde{x} \in \tilde{X}$ there is an open neigborhood $\tilde{U}$ of $\tilde{x}$ such that $g(U) \cup U = \emptyset$ for every $g \in G$.*

An action of a group on a space by continuous maps is called *properly discontinuous* if the above condition is met. Note that any two translates $g(U)$ and $h(U)$ of U by elements of G are disjoint. (Proof?)

PROOF. Let $x = p(\tilde{x})$, choose an admissible open connected neighborhood U of x, and let $\tilde{U}$ be the component of $p^{-1}(U)$ containing $\tilde{x}$. $g(\tilde{U})$ must be a connected component of $p^{-1}(U)$ and since $g(\tilde{x}) \neq \tilde{x}$, it can't be $\tilde{U}$. $\square$

We would now like to be able to reverse this reasoning.

PROPOSITION 3.40. *Let G be a group of continuous maps of a connected, locally path connected space $\tilde{X}$, and suppose the action is properly discontinuous. Then the quotient map $p : \tilde{X} \to X = \tilde{X}/G$ is a regular covering, and $\mathrm{Cov}_X(\tilde{X}) = G$.*

PROOF. First note that p is an open map. For, let $\tilde{U}$ be an open set in $\tilde{X}$. Then, since each $g \in G$ is in fact a homeomorphism (which follows from the defintion of the action of a group on a space), it follows that each $g(\tilde{U})$ is open. Hence,

$$p^{-1}(p(\tilde{U})) = \bigcup_{g \in G} g(\tilde{U})$$

is open. So, by the definition of the topology in the quotient space, $p(\tilde{U})$ is open.

It now follows that p is a covering. For, given $x \in X$, pick $\tilde{x}$ lying over x and let $\tilde{U}$ be an open neighborhood of $\tilde{x}$ such that the open sets $g(\tilde{U})$ for $g \in G$ are all disjoint. Since p is an open map, $U = p(\tilde{U})$ is an open neighborhood of x and $p^{-1}(\tilde{U}) = \cup_g g(\tilde{U})$ is a disjoint union of open sets. Moreover, by the definition of the set $\tilde{X}/G$, $p|\tilde{U}$ is certainly one-to-one and onto, and since it is continuous and open, it is a homeomorphism. Finally, since $p \circ g = p$, the same is true for $p|g(\tilde{U}) \to U$ for any $g \in G$.

To show that $G = \mathrm{Cov}_X(\tilde{X})$, first note that G is a group of covering maps over X, so $G \subseteq \mathrm{Cov}_X(\tilde{X})$. To see that they are equal, consider $f(\tilde{x}) \in p^{-1}(x)$ for $f \in \mathrm{Cov}_X(\tilde{X})$. The fiber is just an orbit under the action of G, so

$$f(\tilde{x}) = g(\tilde{x})$$

for an appropriate $g \in G$. By the uniqueness lemma, $f = g$.

Finally, we show that the covering is normal (regular). Let $\tilde{x}, \tilde{y} = \tilde{x}\alpha$ be arbitrary points in $p^{-1}(x)$. Then, since as above, $\tilde{y} = g(\tilde{x})$ for some covering map g, it follows that $p_*(\pi(\tilde{X}, \tilde{x})) \subseteq p_*(\pi(\tilde{X}, \tilde{y})) = \alpha^{-1}p_*(\pi(\tilde{X}, \tilde{x})\alpha$. Since α is arbitrary, it is easy to see that $p_*(\pi(\tilde{X}, \tilde{x}))$ is normal as required. $\square$

Note that is is fairly clear that the action of $\mathrm{Cov}_X(\tilde{X})$ on $\tilde{X}$ is properly discontinous for any covering $\tilde{X} \to X$. It is natural to ask then if the quotient space for this action is X again. In fact, this will happen only in the case that the covering is regular. We leave it to the student to check that this is true.

Suppose now that $p : \tilde{X} \to X$ is a universal covering where X is connected and locally path connected. Let H be any subgroup of $\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)$. H certainly acts properly discontinuously on $\tilde{X}$, so we may let $\tilde{X}' = \tilde{X}/H$. Then, the quotient map $q : \tilde{X} \to \tilde{X}'$ is a regular covering with covering group H. (See the Exercises.) Define $p' : \tilde{X}' \to X$ by $r(\tilde{x}') = p(\tilde{x})$ where $\tilde{x} \in q^{-1}(\tilde{x}')$. Since all such $\tilde{x}$ are related by elements of H, they are in a single fiber for p, so they project to the same element $p(\tilde{x})$. Thus, p' is well defined. We leave it to the student to show that p' is continuous and a covering map.

Lemma 3.41. *With the above notation, $q : \tilde{X} \to \tilde{X}'$ is consistent with the action of $\pi(X, x)$ on fibers, i.e.,*

$$q(\tilde{x}\alpha) = q(\tilde{x})\alpha$$

for $\tilde{x} \in \tilde{X}$ and $\alpha \in \pi(X, p(\tilde{x}))$.

Proof. Use the fact that $q(\tilde{x})$ is the orbit $H\tilde{x}$ and that the actions of $\mathrm{Cov}_X(\tilde{X})$ and $\pi(X, x)$ on $p^{-1}(x)$ (where $x = p(\tilde{x})$) are consistent. $\square$

Proposition 3.42. *Let X be connected and locally path connected and suppose X has a universal covering $p : \tilde{X} \to X$. Let $x \in X$. Then every subgroup H' of $\pi(X, x)$ is of the form $p'_*(\pi(\tilde{X}', \tilde{x}'))$ for some covering $p' : \tilde{X}' \to X$.*

What this proposition tells us, together with what was proved before, is that there is a one-to-one correspondence between the collection of isomorphism classes of coverings with base points and the collection of subgroups of the fundamental group of X. Similarly, there is a one-to-one correspondence between the collection of isomorphism classes of coverings (ignoring base points) and the collection of conjugacy classes of subgroups of the fundamental group.

Proof. An isomorphism $\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)$ may be specified as follows. Let $p(\tilde{x}) = x$. Then

$$g \leftrightarrow \alpha \qquad \text{if and only} \qquad g(\tilde{x}) = \tilde{x}\alpha.$$

Let H correspond to H' under this isomorphism, and consider the covering $p' : \tilde{X}' = \tilde{X}/H \to X$ as above. We have

$$\begin{aligned} \tilde{x}'\alpha = \tilde{x}' &\Leftrightarrow q(\tilde{x}\alpha) = q(\tilde{x})\alpha = q(tx) \\ &\Leftrightarrow \tilde{x}\alpha = h(\tilde{x}) \quad \text{for some } h \in H. \end{aligned}$$

However, this just says that α fixes $\tilde{x}'$ if and only if it corresponds under the isomorphism to an element of H, i.e., if and only if it is an element of H'. Hence,

$$H' = p'_*(\pi(\tilde{X}', \tilde{x}'))$$

as claimed. □

Example 3.43. Consider the universal covering $E^n : \mathbf{R}^n \to T^n$. Fix a point $x \in T^n$. The fundamental group $\pi(T^n, x) \cong \mathbf{Z}^n$ as mentioned earlier. Also, $\mathrm{Cov}_{T^n}(\mathbf{R}^n)$ consists of all translations of $\mathbf{R}^n$ by vectors $\mathbf{k} = (k_1, \dots, k_n)$ with integral components. Hence, we can determine all coverings of T^n by describing all subgroups of $\mathbf{Z}^n$.

First consider the case $n = 1$. Then every non-trivial subgroup of $\mathbf{Z}$ is of the form $H = m\mathbf{Z}$ for some positive integer m. The corresponding covering space $\tilde{X}' = \mathbf{R}/H$ is homemorphic to S^1 again, but where instead of identifying points in $\mathbf{R}$ which are 1 unit apart, we identify points which are m units apart.

The case $n > 1$ is similar but the algebra is more complicated. An example for $n = 2$ is indicated diagramatically below.

4.2. Existence of Universal Covering Spaces.

THEOREM 3.44. *Let X be a connected, locally path connected space. Then, X has a universal covering space if and only if it satisfies the following property: for each point $x \in X$ there is an open neighborhood U of x such that every loop in U at x is homotopic in X (relative to $\dot{I}$) to the constant loop x.*

This property has the confusing name 'semi-locally simply connected.'

You can find the proof of this theorem in Massey and elsewhere.

5. Covering Groups

Let X be a connected, locally connected space, and suppose it has a universal covering space $\tilde{X}$. Suppose in addition that X has the structure of a *topological group*, i.e., it is a group in which the group operation and inverse map are continuous. Then, it is possible to show that $\tilde{X}$ may also be endowed with the structure of a topological group such that the covering map $p : \tilde{X} \to X$ is a group epimorphism. (See Massey—which has some exercises with hints to prove this—or one of the other references on covering spaces.) In this case, it is easy to see that we can identify the kernel K of the homomorphism p with $\mathrm{Cov}_X(\tilde{X}) \cong \pi(X, x)$. For, if $\tilde{k} \in K$, then the map $f_{\tilde{k}}$ defined by

$$f_{\tilde{k}}(\tilde{x}) = \tilde{K}\tilde{x}$$

is easily seen to be a covering map. Since $f_{\tilde{k}}(\tilde{1}) - \tilde{k}$, $\tilde{k} \mapsto f_{\tilde{k}}$ is a monomorphism. Also, by group theory, the fiber over any point $p^{-1}(x)$ is just the coset $\tilde{x}K = K\tilde{x}$ for any $\tilde{x} \in p^{-1}(x)$. If $f \in \mathrm{Cov}_X(\tilde{X})$, then there is a $\tilde{k} \in K$ such that $f(\tilde{x}) = \tilde{k}\tilde{x} = f_{\tilde{k}}(\tilde{x})$ so $f = f_{\tilde{k}}$. Hence, $\tilde{k} \mapsto f_{\tilde{k}}$ is an isomorphism.

EXAMPLE 3.45 (Torii). Consider $E^n : \mathbf{R}^n \to T^n$. The kernel is $\mathbf{Z}^n$ and this is the covering group as mentioned before.

EXAMPLE 3.46 (The rotation group). First consider the group $Gl(n\mathbf{R})$ of all $n \times n$ invertible matrices. Viewed as a subset of $\mathbf{R}^{n^2}$ it becomes a topological group. It is not connected, but if we take the collection of all invertible matrices with positive determinant, then this is path connected. (See the Exercises.) This collection of matrices is the path component containing the identity and it is a normal subgroup S of $Gl(n, \mathbf{R})$. There is one other component, namely RS where R is the matrix with -1 in the $1, 1$ position and is otherwise the same as the identity matrix. (R represents a reflection. Any other reflection would do as well.) These facts are tied up with the idea of orientation in $\mathbf{R}^n$. Namely, all possible bases are divided into two classes. Those determined by coordinate transformation matrices with positive determinant and those determined by coordinate transformations with negative determinant. These are called the orientation classes of the bases, and a reflection switches orientation.

Consider in particular the case $n = 3$ and consider the subgroup $O(3)$ of $Gl(3, \mathbf{R})$ consisting of all orthogonal 3×3 matrices. Since the determinant of an orthogonal matrix is ± 1, $O(3) \cap S$ consists of all 3×3 orthogonal matrices of determinant 1, and is denoted $SO(3)$. $SO(3)$ is also path connected. To see this among other things, we shall construct a surjective map $p : S^3 \to SO(3)$ with the property that the inverse image of every point in $SO(3)$ is a pair of antipodal points in S^3. It follows from the existence of such a map that $SO(3)$ is homeomorphic to $\mathbf{R}P^3$ and that $p : S^3 \to SO(3)$ is a universal covering. By what we noted above, S^3 must have the structure of a topological group, and $\pi(SO(3), I)$ may be identified with the kernel of the projection p. S^3 with this group structure is denoted $Spin(3)$, and the kernel is cyclic of order two. (The only S^n which can be made into topological groups are S^1 and S^3.)

To define the map p, first consider D^3. We shall define a surjection $p : D^3 \to SO(3)$ which is one-to-one on the interior and maps antipodal points on the boundary S^2 to the same point. (Then as before, we may produce from this a quotient map from S^3 to $SO(3)$ which sends antipodal points to the same point by using the usual relation between the upper 'hemisphere' of S^3 and D^3.) $p : D^3 \to SO(3)$ is defined as follows. Let $p(0) = I$. For $x \neq 0 \in D^3$, let $p(x)$ be the rotation of $\mathbf{R}^3$ about the axis x and through the angle $|x|\pi$ (using the right hand rule to determine the direction to rotate). The matrix of the rotation $p(x)$ with respect to an appropriate coordinate system has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta = |x|\pi$, so it is orthogonal with determinant 1. The only ambiguity in describing $p(x)$ is that rotation about the axes x and $-x$ through angle π are the same, which is to say that the mapping is one-to-one except for points with $|x| = 1$ where it maps each pair of antipodal points to the one point.

We leave it to the student to verify that p is a continous map.

The only thing remaining is to show that p is onto. We do this by showing that every non-trivial element of $SO(3)$ is a rotation. First note that any orthogonal matrix 3×3 matrix A has at least one real eigenvalue since its characteristic equation is a cubic. The absolute value of that eigenvalue must be 1, Since the product of the complex eigenvalues of A is $\det A = 1$, it is not hard to see that at least one eigenvalue must be 1. This says the correpsonding eigenvector v is in fact fixed by A. Change to an orthonormal bais with v the first basis vector. With respect to this basis, A has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where A' is a 2×2 orthogonal matrix of determinant 1. However, it is not hard to see that any such matrix must be of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

i.e., it is a 2×2 rotation matrix. This in fact shows that A is the rotation about the axis v through angle θ, and we may certainly assume $0 \leq \theta \leq \pi$.

CHAPTER 4

Group Theory and the Seifert–Van Kampen Theorem

1. Some Group Theory

We shall discuss some topics in group theory closely connected with the theory of fundamental groups.

1.1. Finitely Generated Abelian Groups. We start with the followng theorem which you may have seen proved in a course in algebra either at the graduate or undergraduate level. We leave the proof for such a course.

THEOREM 4.1 (Structure Theorem for Finitely Generated Abelian Groups). *A finitely generated abelian group A is isomrphic to a direct sum of cyclic groups.*

Associated with this theorem is a uniqueness result, the statement of which requires some terminology. Note first that in discussing abelian groups one often uses additive notation and terminology and we shall generally, but not always, do that here. In the decomposition asserted in the theorem, some of the factors will be finite cyclic groups and some will be isomorphic to the infinite cyclic group $\mathbf{Z}$. The finite cyclic factors add up to a subgroup $t(A)$ of A which consists of all elements of A of finite order. This is called the torsion subgroup of A. The quotient group $A/t(A) = f(A)$ has no torsion elements, so it is a direct sum of copies of $\mathbf{Z}$ or what we call a *free abelian group*. The theorem then can be read as asserting that the projection $p : A \to f(A)$ has a left inverse $r : f(A) \to A$ whence

$$A = t(A) \oplus r(f(A)) \cong t(A) \oplus f(A).$$

In language you are probably familiar with by now, this is summarized by saying that the short exact sequence

$$0 \to t(A) \to A \to f(A) \to 0$$

splits. (A sequence of groups and homomorphisms is exact if at each stage, the image of the homomorphism into a group equals the kernel of the homomorphism out of it. For a sequence of the above type, exactness at $t(A)$ says the homomorphism $i : t(A) \to A$ is a monomorphism and exactness at $f(A)$ says $p : A \to f(A)$ is an epimorphism. Exactness at A says $A/\operatorname{Im} i \cong f(A)$.) Note that the torsion subgroup $t(A)$ is uniquely determined by A but the complementary subgroup $r(f(A)) \cong f(A)$ depends on the choice of r.

As note above, the group $f(A)$ is a free abelian group, i.e., it is a direct sum of copies of $\mathbf{Z}$. The number of copies is called the *rank* of A. The rank of a free abelian group may be defined using the concept of basis as in the theory of vector spaces over a field. As in that theory one shows that every basis has the same number of elements which is the rank. That gives us one part of the uniqueness statement we want. The discussion of the subgroup $t(A)$ is a bit more complicated. It is necessarily a direct sum of a finite number of finite cyclic groups, but there can be many different ways of doing that depending on the orders of those cyclic groups. For example.

$$\mathbf{Z}/6\mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$$

but

$$\mathbf{Z}/4\mathbf{Z} \ncong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

THEOREM 4.2 (Uniqueness Theorem for Finitely Generated Abelian Groups). *Let A be a finitely generated abelian group and suppose*

$$\begin{aligned} A &\cong \mathbf{Z}/d_1\mathbf{Z} \oplus \mathbf{Z}/d_2\mathbf{Z} \oplus \dots \mathbf{Z}/d_k\mathbf{Z} \oplus M \\ &\cong \mathbf{Z}/d'_1\mathbf{Z} \oplus \mathbf{Z}/d'_2\mathbf{Z} \oplus \dots \mathbf{Z}/d'_{k'}\mathbf{Z} \oplus M' \end{aligned}$$

where M, M' are free, and $d_1|d_2|\dots|d_k$, $d'_1|d'_2|\dots|d'_{k'}$ are integers > 1. Then M and M' have the same rank (so are isomorphic), $k' = k$ and $d'_i = d_i, i = 1, \dots, k$.

As above, we assume this was proved for you (or will be proved for you) in an algebra course.

An alternate approach to the uniqueness statement first divides $t(A)$ up as a direct sum of its Sylow subgroups. Each Sylow group has order p^N for some prime p, and if we break it up as a direct sum of cyclic subgroups, we can arrange the divisibility relations simply by writing the factors in increasing order. This gives an a somwhat different version of the theorem where the orders of the cyclic factors are unique prime powers (whether in order or not), but we eschew simplifications of the form $\mathbf{Z}/12\mathbf{Z} \cong \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$.

One important property of a free abelian group M is that any homomorphism of abelian groups $f : M \to N$ is completely determined by what it does to a basis, and conversely, we may always define a homomorphism by specifying it on a basis and extending it by linearity to

M. (This generalizes the corresponding statment about vector spaces, linear transformations, and bases.) The may be restated as follows. Let X be a basis for M and let $i : X \to M$ be the inclusion map. Then $M, i : X \to M$ has the following *universal mapping property*. Let $j : X \to N$ be any mapping of X to an abelian group N (as sets). Then there exists a unique homomorphism $f : M \to N$ such that $j = f \circ i$.

1.2. Free Groups. Let X be a set. We want to form a group $F(X)$ which is generated by the elements of X and such that there are as 'few' relations among the elements of X as possible consistent with their being elements of a group. We shall outline the construction of such a group here. (A complete treatment of this is done in algebra courses. There is also a discussion in Massey.) First consider all possible sequences—called words—of the form

$$x_1{}^{e_1} x_2{}^{e_2} \dots x_n{}^{e_n}$$

where each $x_i \in X$ and $e_i = \pm 1$. Include as a possibility the empty word which we shall just denote by 1. Then justaposition defines a law of composition on the set of all such words. Define a relation on the set of all such words as follows. Say $w \sim w'$ if w contains a two element subsequence of the form xx^{-1} or $x^{-1}x$ and w' is the word obtained by deleting it *or vice-versa*. Call this an elementary reduction. In general say that $w \sim w'$ if there is a sequence of elementary reductions $w = w_1 \sim w_2 \sim \cdots \sim w_k = w'$. It is not hard to check that this is an equivalence relation and that it is consistent with the binary operation. That is, if $w_1 \sim w_1'$ and $w_2 \sim w_2'$ then $w_1 w_2 \sim w_1' w_2'$. An important fact useful in analyzing this relation is that each equivalence class of words contains a unique word of minimal length (containing no xx^{-1} or $x^{-1}x$) called a reduced word.

It follows that the set of equivalence classes $F(X)$ is endowed with a binary operation. With some work, it is possible to see that it is a group, called the free group on the set X. $F(X)$ has the following universal mapping property. Let $i : X \to F(X)$ be the map of sets defined by letting $i(x)$ be the equivalence class of the word x. Let $j : X \to G$ be a set map from X to a group G. Then there is a unique group homomoprhism $f : F(X) \to G$ such that $j = f \circ i$.

EXAMPLE 4.3. The free group $F(x)$ on one generator x is infinite cyclic. If G is any cyclic group generated say by a of order n, then $x \mapsto a$ defines a homomorphism $f : F \to G$. This is an epimorphism and its kernel is the cyclic subgroup R of F generated by x^n. Clearly, $F/R \cong G$.

EXAMPLE 4.4. Let $F = F(x, y)$ and let G be the dihedral group of order 8. G is generated by a, b where $a^4 = 1, b^2 = 1$ and $bab^{-1} = a^{-1}$. Then $f : F \to G$ is defined by $x \mapsto a, y \mapsto b$. The kernel of f contains the elements $x^4, y^2, yxy^{-1}y$. Let R be the normal subgroup of F generated by these elements, i.e., the smallest normal subgroup containing them. R consists of all possible products of *conjugates in* F of these three elements and their inverses. By doing some calculations with elements, it is possible to show that $|F/R| = 8$. Since $F \to G$ is clearly onto, it follows that f induces an isomorphism $F/R \cong G$.

EXAMPLE 4.5. Let $F = F(x, y)$ and let $G = \mathbf{Z} \times \mathbf{Z}$ where this time we write G multiplicatively and denote generating basis elements by a and b. Define $f : F \to G$ by $x \mapsto a, y \mapsto b$. Then it is possible to show that $R = \operatorname{Ker} f$ is the normal subgroup generated by $xyx^{-1}y^{-1}$. Note that the word 'normal' is crucial. Indeed, it can be shown that any subgroup of a free group is free, so R is free, but viewed as a group in its own right it has denumerably many generators.

The above examples illustrate a very general process. For any group G generated say by a set X, we can a free group $F(X')$ where $|X'| = |X|$ and an epimorphism $f : F(X') \to G$. In fact, if one is careful with the notation, there is no point in not taking $X' = X$. If the kernel R of this epimorphism is generated as a normal subgroup by elements $r_1, r_2, \dots$, we say that we have a *presentation* of G as the group generated by the elements of X subject to the relations $r_1, r_2, \dots$. This in principle reduces the study of G to the study of F/R. However, in most cases, it is hard to determine when two words determine the same element of the quotient. (It has in fact been proved that there can be no general algorithm which accomplishes this!)

1.3. Free Products. We may generalize the construction of the free group as follows. Let H, K be two groups. We form a group $H * K$ called the free product of H with K as follows. Consider all words of the form

$$g_1 g_2 \cdots g_n$$

where for each i either $g_i \in H$ or $g_i \in K$. Allow the empty word which is denoted by 1. Then concatenation defines a binary operation on the set of all such words. Define an equivalence relation as before. First, say $w \sim w'$ by an elementary reduction if one of the two words contains a subsequence $g_i g_{i+1}$ where both are in H or both are in K and the other has length one less and contains instead the the single element of H or of K which is their product. (If $g_i = {g_{i+1}}^{-1}$ then assume the other has length two less and no corresponding element.) Say two words are equivalent in general if there is a sequence of elementary reductions going from one to the other. It is possible to show that any equivalence class contains a unique reduced word of the form

$$h_1 k_1 h_2 k_2 \dots h_r k_r$$

where each $h_i \in H$, each $k_i \in K$, all the $h_i \neq 1$ except possibly the first, and all the $k_i \neq 1$ except possibly for the last. As above, the equivalence relation is consistent with concatenation and defines a binary operation on the set $H * K$ of equivalence classes which is called the *free product* of K with H.

The free product has the following universal mapping property. Let $i : H \to H * K$ be the map defined by letting $i(h)$ be the equivalence of the word h and similarly for $j : K \to H * K$. Then given any pair of homomorphisms $p : H \to G, q : K \to G$, there is a unique homomorphism $f : H * K \to G$ such that $p = f \circ i, q = f \circ j$.

Namely, we define

$$f(g_1 g_2 \dots g_n) = g_1' g_2' \dots g_n'$$

where $g_i' = p(g_i)$ if $g_i \in H$ and $g_i' = q(g_i)$ if $g_i \in K$. For a reduced word,

$$f(h_1 k_1 \dots h_l k_l) = p(h_1) q(k_1) \dots p(h_l) q(k_l).$$

With some effort, one may show that f sends equivalent words to the same element and that it defines a homomoprhism.

The universal mapping property specifies $H * K$ up to isomorphism. For, suppose P' were any group for which there were maps $i' : H \to P', j' : K \to P'$ satisfying this same universal mapping property. Then there exist homomorphisms $f : H * K = P \to P'$ and $f' : P' \to P = H * K$ with appropriate consistency properties

The homomorphisms $f \circ f'$ and $f' \circ f$ have the appropriate consistency properties for homomorphisms $P' \to P'$ and $P \to P$ as do the identity homomorphisms. Hence, by uniqueness, $f \circ f'$ and $f' \circ f$ are the identities of P' and P respectively, whence f and f' are inverse isomorphisms.

This same construction can be repeated with any number of groups to form the free product $H_1 * H_2 * \cdots * H_l$, which has a universal mapping property you should state for yourself. Indeed,

$$H_1 * (H_2 * \cdots * H_l) \cong H_1 * H_2 * \cdots * H_l.$$

The free product of groups is related to the concept of free group as follows. Let $X = \{x_1, x_2 \ldots, x_n\}$, and let $F_i = F(x_i)$ be the infinite cyclic group which is the free group generated by x_i. Then

$$F(X) \cong F_1 * F_2 * \cdots * F_n.$$

Since all the F_i are isomorphic to $\mathbf{Z}$, we may also write this

$$F(X) \cong \mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z} \quad n \quad \text{times}.$$

1.4. Free Products with Amalgamation. Let H and K be groups and suppose we have two homomorphisms $i : A \to H$ and $j : A \to K$. We want to define something like the free product but where elements of the form $i(a)$ and $j(a)$ are identified. To this end, consider the normal subgroup N of $H * K$ generated by all elements of the form

$$i(a)j(a)^{-1} \qquad \text{for } a \in A.$$

We denote the quotient group $H * K/N$ by $H *_A K$, and call it the free product with amalgamation. (The notation has some defects since the group depends not only on A but also on the homomorphisms i and j. This concept was first studied in the case A is a subgroup of both H and of K.)

A general element of $H *_A K$ can be written

$$g_1 g_2 \cdots g_n$$

where either $g_i \in H$ or $g_i \in K$. In addition to equalities which result from equivalences of words in $H * K$, we have for each $a \in A$, the rule

$$i(a) = j(a)$$

in $H *_A K$. Note that we have decided to engage in a certain amount of 'abuse of notation' in order to keep the notaion relatively simple. An element of $H * K$ is actually an equivalence class of words, and a element of $H *_A K$ is a coset of the normal subroup N determined by an element of $H * K$. Hence, when we assert that an element of $H *_A K$ is of the form

$$g_1 g_2 \dots g_n$$

where either $g_i \in H$ and $g_i \in K$, we are ignoring several levels of abstraction.

The free product with amalgamation may also be characterized by a universal mapping property. Define $p : H \to H *_A K$ first mapping h to the word h in $H * K$ and then to the coset of h in $H * K/N$. Similarly, define $q : K \to H *_A K$. Then we have a commutative diagram

Suppose generally, we have $p' : H \to G$ and $q' : K \to G$ such that $p' \circ i = q' \circ j$, i.e.,

By the universal mapping property of $H * K$, there exists a unique map $f' : H * K \to G$ such that

$$f'(\underbrace{h_1 k_1 \dots h_l k_l}_{\text{in } H*K}) = p'(h_1)q'(k_1) \dots p'(h_l)q'(k_l).$$

In particular, for the element $i(a)j(a)^{-1} \in H * K$, we have

$$f'(\underbrace{i(a)j(a)^{-1}}_{\text{in } H*K}) = p'(i(a))q'(j(a))^{-1} = 1$$

so the normal subroup of $H * K$ generated by all such elements is contained in $Ker f'$. Hence, f' induces a homomorphism $f : H * K/N = H *_A K \to G$ making the following diagram commute.

On the other hand, there can be at most one such homomorphism, since $H *_A K$ is generated by the cosets of words of the form $h_1 k_1 \dots h_l k_l$, so the commutativity of the diagram assures us that

$$f(\underbrace{h_1k_1 \dots h_lk_l}_{\text{in } H *_A K}) = p'(h_1)q'(k_1) \dots p'(h_l)q'(k_l).$$

The argument above was made a bit more confusing than it needs to be by the ambiguity of the notation, but if you pay careful attention, you should have no trouble following it.

The universal mapping property described above is often said to specify $H *_A K$ as the *pushout* in the diagram

2. The Seifert–Van Kampen Theorem

THEOREM 4.6. *Let X be a path connectedspace and suppose $X = U \cup V$ where $U, V, U \cap V$ are open sets of X which are also path connected. Denote the inclusion maps as indicated below*

Let $x_0 \in U \cap V$. Then

presents $\pi(X, x_0)$ as a pushout, i.e.,

$$\pi(X, x_0) \cong \pi(U, x_0) *_{\pi(U \cap V, x_0)} \pi(V, x_0).$$

We defer the proof until later.

EXAMPLE 4.7 (The Figure eight). Let X be the figure eight space described earlier. Let U be the subspace obtained by deleting the closed half of one loop and let V be the subspace obtained by doing the same with the other loop.

Then U and V are each deformation retracts of spaces homeomorphic to S^1 and $U \cap V$ is contractible. Hence,

$$\pi(X) \cong \pi(U) *_1 \pi(V) \cong \mathbf{Z} * \mathbf{Z}$$

so it is free on two generators. Indeed, checking the details of the contruction, we may take as generators the generators of the fundamental groups of the two loops.

Example 4.8 (The 2-torus). We shall show yet again that $\pi(T^2) \cong \mathbf{Z} \times \mathbf{Z}$. Realize T^2 as a square with opposite edges identified.

Let U be the open square without boundary, and let V be the open set obtained by deleting the center point. It is not hard to see that the boundary is homeomorphic to the figure eight space and that it is a deformation retract of V. Hence, $\pi(V) \cong \mathbf{Z} * \mathbf{Z}$ is free on two generators $[a]$ and $[b]$, each representing a circle in T^2. U is contractible so $\pi(U) = \{1\}$. Finally, $U \cap V$ is a punctured disk, so it has the homotopy type of S^1 and $\pi(U \cap V) \cong \mathbf{Z}$. Better yet, $\pi(U \cap V)$ is generated by any simple loop enclosing the deleted center point, so we may choose in particular a square h as indicated in the diagram.

It is clear that h is homotopic in V to the path $a * b * \overline{a} * \overline{b}$. Hence, the pushout diagram describing $\pi(T^2)$ becomes

It follows without two much diffculty that $\pi(T^2)$ is the free group on two generators $\alpha = [a], \beta = [b]$ modulo the normal subgroup generated by the relation $\alpha\beta\alpha^{-1}\beta^{-1}$. It follows that $\pi(T^2)$ is free abelian of rank two.

PROOF. Since we use the common base point x_0 for all fundamental groups, we shall omit it from the notation.

By the universal mapping property of the pushout, we know there is a unique homomorphism $\pi(U) *_{\pi(U\cap V)} \pi(V) \to \pi(X)$ making the appropriate diagram commute, and we can even describe it explicitly on a general element in the free product with amalgamation. (See the previous section.) We already proved (when showing that S^n is simply connected for $n > 1$) that that any loop in X at x_0 is homotopic reltative to $\dot{I}$ to a product of loops each of which is either in U or is in V. Translation of this asserts that $\pi(X)$ is generated by products of the form

$$\gamma_1\gamma_2 \dots \gamma_k$$

where for each i, either $\gamma_i = p_*(\alpha_i), \alpha_i \in \pi(U)$ or $\gamma_i = q_*(\alpha_i), \alpha_i \in \pi(V)$. It follows that the homomorphism $G = \pi(U) *_{\pi(U\cap V)} \pi(V) \to \pi(X)$ is an epimorphism.

The hard work of the proof is to show that the homomorphism is a monomphism.

Let

$$h_1 * h_2 * \dots * h_k \sim_{\dot{I}} e_{x_0}$$

in X where $h_1, \dots, h_k$ are loops at x_0, each of which is either in U or in V. Let U_i be either U or V accordingly, and let $\alpha_i = [h_i]_{U_i}$ denote the class of h_i in $\pi(U_i)$. We want to show that the correponding element

$$\underbrace{\alpha_1\alpha_2 \dots \alpha_k = 1}_{\text{in } \pi(U) *_{\pi(U\cap V} \pi(V)}.$$

Choose $h : I \to X$ such that $h \sim_{\dot{I}} h_1 * h_1 * \dots * h_k * h_k$ in X by subdividing I into k subintervals such that the restriction of h to the ith subinterval is h_i after a suitable parameter change. Let $H : I \times I \to X$ be a homotopy realizing $h \sim_{\dot{I}} e_{x_0}$, i.e.,

$$\begin{aligned} H(t,0) &= h(t) \\ H(t,1) &= x_0 \\ H(0,s) &= H(1,s) = x_0. \end{aligned}$$

By the Lebesgue Covering Lemma, we may choose partitions

$$\begin{aligned} 0 = t_0 < t_1 < \dots < t_n = 1 \\ 0 = s_0 < s_1 < \dots < s_m = 1 \end{aligned}$$

which subdivide $I \times I$ into closed subrectangles $R_{ij} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$ such that for each i, j, either $H(R_{ij}) \subseteq U$ or $H(R_{ij}) \subseteq V$.

As above, let U_{ij} be U or V accordingly. We can also arrange the partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

to include the endpoints of the domains of the functions h_r. Just throw those points in as needed; the rectangles in the subdivision may become smaller. Now redefine

$$h_i = h|[t_{i-1}, t_i] \qquad i = 1, 2, \dots, n.$$

This does not affect the equivalence

$$h_1 * h_2 * \cdots * h_n \sim_{\dot{I}} e_{x_0}$$

in X, but it does create one technical problem. Namely, the new paths h_i need not be loops.

Avoid this difficulty as follows. For each point $x \in X$, choose a path g_x from x_0 to x such that the image of g_x lies in U, V, or $U \cap V$ as x itself does. (This is possible since U, V, and $U \cap V$ are path connected.) Also, let g_{x_0} be constant. For each $i = 0, \dots, n$, abbreviate $g_i = g_{h(t_i)}$. Then, $g_{i-1} * h_i * \overline{g}_i$ is a loop, and we shall abbreviate

$$[h_i] = [g_{i-1} * h_i * \overline{g}_i].$$

Similar conventions will be needed for the other vertices in the subdivision of $I \times I$ into rectangles. We shall be interested in paths in $X, U, V, U \cap V$ obtained by restricting H to the edges of these rectangles. These won't generally be loops at x_0. However, we may choose paths as above from x_0 to the endpoints of these edges and reinterpet the notation '$[-]$' accordingly. These base point shifting paths should be chosen so they lie in U, V, or $U \cap V$ if their endpoints do, and they should be constant if the endpoint happens to be x_0.

Consider paths in X defined by restricting H to a rectilinear path in $I \times I$ which starts at the left edge, follows a sequence of horizontal edges of the rectangles R_ij, perhaps drops down through one vertical edge, and then continues horizontally to the right hand edge.

This will be a loop in X based at x_0 since H takes on the value x_0 on both left and right edges. We may proceed by a sequence of such paths from h on the bottom edge of $I \times I$ to e_{x_0} on the top edge. The elementary step in this sequence is to 'add' a rectangle as indicated in the diagram. Consider the effect of adding the rectangle R_{ij}. Let h_{ij} be the path associated with the horizontal edge from (t_i, s_{j-1}) to (t_i, s_j) and let v_{ij} be the path associated with the vertical edge from (t_i, s_j) to (t_{i-1}, s_j). Then, in U_{ij} and in X we have

$$h_{ij} * v_{ij} \sim_{\dot{I}} v_{i,j-1} * h_{i-1,j}.$$

This has two consequences. First, the effect of adding the rectangle is to produce an equivalent path in X. Second,

$$[h_{ij}][v_{ij}] = [v_{i,j-1}][h_{i-1,j}]$$

is a true in equation in $\pi(U_{ij})$. We should like to conclude that it is also a true equation in $\pi(U) *_{\pi(U\cap V)} \pi(V)$. If that is true, then we can conclude that adding the rectangle changes the word associated with one path to the word associated to the other in $\pi(U) *_{\pi(U\cap V)} \pi(V)$. However, there is one problem associated with this. A path h_{ij} or v_{ij} may appear in the free product with amalgamation as coming from either of the two rectangles which it abuts. Hence, the path could lie in both U and in V and be considered to yield both an element in $\pi(U)$ and an element of $\pi(V)$. However, in the free product with amalgamation, these elements are the same because they come from a common element of $\pi(U \cap V)$.

It now follows that in $\pi(U) *_{\pi(U\cap v)} \pi(V)$, we have a sequence of equalities

$$[h_1][h_2]\dots[h_n] = \dots = [e_{x_0}]^n = 1.$$

□

We do a couple more examples.

Example 4.9 (The Klein Bottle). As in the example of the 2-torus, the boundary is homeomorphic to a figure eight. The decomposition $X = U \cup V$ is the same as in that case, but the path h is homotopic instead to the path representing $\alpha\beta\alpha^{-1}\beta$. Thus $\pi(X)$ is isomorphic the the free group on two generators α, β modulo the relation $\alpha\beta\alpha^{-1} = \beta^{-1}$. Let B be the subgroup generated by β. It is not hard to see it is normal and the quotient is generated by the coset of α. However, for any n, m we can contruct a finite group of order nm generated by elements a, b and such that $a^n = b^m = 1$ and $aba^{-1} = b^{-1}$. (You can actually write out a multiplication table for such a group and check laboriously that it is a group.) This group is an epimorphic image of the above group. Hence, it follows that the subgroup B must be infinite cyclic as must the quotient group. In fact, the group under consideration is an example of what is called a semi-direct product.

Example 4.10 (The Real Projective Plane). Use the square with the identification below.

Apply exactly the same reasoning as for the Klein bottle and T^2. However, in this case the bounding edges form a space homomorphic to S^1 (divided into two parts). The generator of $\pi(V)$ is $[ab]$ and the path h is homotopic to $[ab]^2$. Hence, $\pi(X)$ is isomorphic to the groups generated by $\gamma = [ab]$ and subject to the relation $\gamma^2 = 1$. Thus it is cyclic of order two.

CHAPTER 5

Manifolds and Surfaces

1. Manifolds and Surfaces

Recall that an n-manifold is a Hausdorff space in which every point has a neighborhood homeomorphic to a open ball in $\mathbf{R}^n$.

Here are some examples of manifolds. $\mathbf{R}^n$ is certainly an n-manifold. Also, S^n is an n-manifold. D^n is not a manifold, but there is a more general concept of 'manifold with boundary' of which D^n is an example. The figure eight space is not a manifold because the point common to the two loops has no neighborhood homeomorphic to an open interval in $\mathbf{R}$.

Since the product of an n-ball and an m-ball is homeomorphic to an $n+m$-ball, it follows that the product of an n-manifold with an m-manifold is an $n+m$-manifold. It follows that any n-torus is an n-manifold.

If $p : \tilde{X} \to X$ is a covering, then it is not hard to see that X is an n-manifold if and only if $\tilde{X}$ is an n-manifold. It follows that $\mathbf{R}P^n$ is an n-manifold. Note that a manifold is certainly locally path connected. Hence, if $\tilde{X}$ is a connected manifold on which a finite group G of homeomorphisms acts, then $\tilde{X} \to X = \tilde{X}/G$ is a covering and X is also a manifold.

In general, a quotient space of a manifold need not be a manifold. For example, a figure eight space may be viewed as a quotient space of S^1. However, many quotient spaces are in fact manifolds. For example, the Klein bottle K is a 2-manifold, as the diagram below indicates.

In fact, if we take any polygon in the plane with an even number of sides and identify sides in pairs we obtain a 2-manifold.

Sometimes one requires that a manifold also be second countable, i.e., that there be a countable set $\mathcal{S}$ of open subsets such that any open set is a union of open sets in $\mathcal{S}$.

It is the purpose of this chapter to classify all compact, connected 2-manifolds. Such manifolds are called *compact surfaces*. (These are also sometimes called closed surfaces.)

One tool we shall use is the concept of *connected sum*. Let S_1, S_2 be surfaces. Choose subspaces $D_1 \subset S_1, D_2 \subset S_2$ which are homeomorphic to disks. In particular, D_1 and D_2 have boundaries $\partial D_1, \partial D_2$ which are homeomorhpic to S^1 and to each other. Suppose we choose a homeomorphism $h : \partial D_1 \to \partial D_2$. We can use this to define an equivalence relation '$\sim$' on the disjoint union of $S_1 - D_1^\circ$ and $S_2 - D_2^\circ$. All equivalence classes are singleton sets except for pairs of the form $\{x, h(x)\}$ with $x \in \partial D_1$. We denote the quotient space by $S_1 \sharp S_2$ and call it the connected sum. Notice that it is connected, and it is not too hard to see that it is a compact 2-manifold.

It is in fact possible to show that different choices of D_1, D_2 and h yield homeomorphic spaces, so $S_1 \sharp S_2$ in fact depends only on S_1 and S_2 up to homeomorphism.

EXAMPLE 5.1. The connected sum of two projective planes $\mathbf{R}P^2 \sharp \mathbf{R}P^2$ is homeomorphic to a Klein bottle K.

Note that we have chosen in a specially convenient way the disks D_1 and D_2, but that is allowable since the location of the disks is immaterial.

The connected sum has certain reasonable properties. First, it is associative up to homeomorphism, i.e.,

$$S_1 \sharp (S_2 \sharp S_3) \cong (S_1 \sharp S_1) \sharp S_3.$$

Moreover, if S is any surface, then

$$S\sharp S^2 \cong S.$$

This says that we may make the set of all homeomorphism classes of surfaces into what is called a monoid. Connected sums provide an associative binary operation with an identity (S^2). Note that the operation is also commutative.

The basic result about classification of surfaces is the following.

THEOREM 5.2. *Let S be a compact surface. Then S is homeomorphic to one of the following:*

(1) S^2,
(2) *the connected sum of n torii ($n \geq 1$).*
(3) *the connected sum of n real projective planes ($n \geq 1$).*

We shall outline a proof of this theorem in the next section.

In the rest of this section, we shall outline a proof that this classifies compact surfaces up to homeomorphism. In fact, we shall show that no surface in one class can have the same fundamental group as one in another group and that within the same group the fundamental group depends on n. This shows in fact that these surfaces are not even of the same homotopy type.

We start with a description of a connected sum of torii as the quotient of disk (or polygon) with portions of its boundary identified. For two torii, the diagram below indicates how to form the connected sum.

Iterate this for a connected sum of three torii.

The general result is a disk with the boundary divided into $4n$ segments

$$a_1b_1\overline{a}_1\overline{b}_1\ldots a_nb_n\overline{a}_n\overline{b}_n$$

identified in pairs (or equivalently a $4n$-gon). The number n is called the *genus* of the surface. The genus may be visualized as follows. A single torus may be thought of as a sphere with a handle attached.

Each time a torus is added, you can think of it as adding another handle.

Hence, the connected sum of n torii can be thought of as a sphere n handles.

A similar analysis works for the connected sum of projective planes. For two, we get a Klein bottle as before.

For three,

Finally, in general, we get a disk or $2n$-gon with edges identified in pairs

$$a_1a_1a_2a_2\dots a_na_n.$$

Consider the possible fundamental groups. For S^2,

the fundamental group is trivial. For a connected sum of n torii $T\sharp T\sharp\dots\sharp T$,

we may apply the Seifert-VanKampen Theorem as in the previous chapter. The fundamental group Π is the free group on $2n$ generators $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ modulo the relation

$$\alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\dots\alpha_n\beta_n{\alpha_n}^{-1}{\beta_n}^{-1} = 1.$$

Note that since the relation is a product of commutators, $\Pi/[\Pi,\Pi]$ is free abelian on $2n$ generators.

To complete the argument, we want to calculate $\Pi/[\Pi.\Pi]$ for the fundamental group of a connected sum of projective planes. Doing this from the diagram is not quite as helpful. We find that the fundamental group Π is the free group on n generators modulo the relation

$$\alpha_1\alpha_1\dots\alpha_n\alpha_n = 1.$$

It follows that $\Pi/[\Pi/\Pi]$ written additively is free on n generators $x_1, x_2, \dots, x_n$ modulo the relation

$$2x_1 + 2x_2 + \dots + 2x_n = 0.$$

By standard techniques in the theory of abelian groups, it is not hard to show that $\Pi/[\Pi,\Pi] \cong \mathbf{Z}^{n-1} \oplus \mathbf{Z}/2\mathbf{Z}$.

However, we shall use another more geometric approach which exhibits the connected sum of n projective planes differently. Not only does this allow us to calculate $\Pi/[\Pi,\Pi]$ without knowing much about abelian groups, but it also allows us to determine easily the connected sum of any two surfaces.

Proposition 5.3. *$\mathbf{R}P^2\sharp\mathbf{R}P^2\sharp\mathbf{R}P^2 \cong T\sharp\mathbf{R}P^2$.*

Proof. Since $\mathbf{R}P^2\sharp\mathbf{R}P^2 \cong K$, it suffices to show that $K\sharp\mathbf{R}P^2 \cong T\sharp\mathbf{R}P^2$. To see this first note that $\mathbf{R}P^2$ contains a subspace isomorphic to a Moebius band.

We shall show that the connected sums of on one hand a Moebius band with a torus and on the other hand a Moebius band with a Klein bottle are homeomorphic. First, the diagram below shows several equivalent ways to describe the first surface.

Next, the following diagram exhibits several ways to visualize a Klein bottle with a disk cut out.

Finally, paste that into a Moebius band with a disk cut out and transform as follows.

□

COROLLARY 5.4. *The connected sum of n projective planes is homeomorphic with the connected sum of a torus with a projective plane if n is odd or with a Klein bottle if n is even.*

PROOF. Apply the above propostition iteratively until you get either a single projective plane (n odd) or two projective planes, i.e., a Klein bottle, (n even). □

We may now calculate the fundamental group.

For $n = 2m + 1$ odd, we see that Π is free on n generators modulo the relation

$$\alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\dots\alpha_m\beta_m{\alpha_m}^{-1}{\beta_m}^{-1}\gamma\gamma = 1.$$

($2m$ generators from the torii and one from the projective plane.) It follows that $\Pi/[\Pi,\Pi] \cong \mathbf{Z}^{2m} \times \mathbf{Z}/2\mathbf{Z}$.

For $n = 2m$ even, we see that Π is free on n generators modulo the relation

$$\alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\dots\alpha_{m-1}\beta_{m-1}{\alpha_{m-1}}^{-1}{\beta_{m-1}}^{-1}\gamma\delta\gamma^{-1}\delta = 1.$$

(The last part of the relation comes from the Klein bottle.) Modulo $[\Pi,\Pi]$, this relation becomes

$$\gamma\delta\gamma^{-1}\delta = \gamma\delta\gamma^{-1}\delta^{-1}\delta^2 = \delta^2 = 1.$$

Hence, $\Pi/[\Pi,\Pi] \cong \mathbf{Z}^{2m-2} \times \mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \cong \mathbf{Z}^{2m-1} \times \mathbf{Z}/2\mathbf{Z}$.

We may summarize all this by

Theorem 5.5. *Let S be a compact surface. If S is homeomorphic to a connected sum of n 2-torii, then*

$$\pi(S)/[\pi(S),\pi(S)] \cong \mathbf{Z}^{2n}.$$

If S is homemorphic to n projective planes, then

$$\pi(S)/[\pi(S),\pi(S)] \cong \mathbf{Z}^{n-1} \times \mathbf{Z}/2\mathbf{Z}.$$

In particular, these surfaces are all distinguished by their fundamental groups.

Note that the surfaces involving a projective plane (explicitly or implicitly as in the case of a Klein bottle) have a factor of $\mathbf{Z}/2\mathbf{Z}$ in the abelianized fundamental group. We call the surfaces without projective plane components *orientable* and those with such components *non-orientable*. We shall explore the issue of orientablity later in great detail. For the moment, notice that the orientable surfaces can be viewed as spheres with handles imbedded in $\mathbf{R}^3$, and we can assume it is a smooth surface with well defined normals at every point. For such a surface we can distinguish the two 'sides' of the surface, and this is one of the meanings of orientation.

For surfaces having a component which is a projective plane, we can pick out a subspace homeomorphic to a Moebius band. It is hoped that the student is familiar with the problem of assigning sides to a Moebius band and the possibility of reversing orientation by going around the band.

2. Outline of the Proof of the Classification Theorem

We shall give a rather sketchy outline of the argument. See Massey for more (but not all) details.

First, we note that every compact surface S can be *triangulated*. This is actually a rather deep theorem not at all easy to prove. It means the following. We can decompose S as a union $\cup_{i=1}^{n} T_i$ of finitely many curvilinear 'triangles' satisfying certain conditions.

That is, each T_i is homeomorphic to an actual closed triangle in $\mathbf{R}^2$. Through the homeomorphisms, it makes sense to discuss the 'vertices' and the 'edges' of the T_i. Then, any two curvilinear triangles T_i and T_j are either disjoint, have a single edge in common, or have a single vertex in common. In addition, each edge is common to exactly two curvilinear triangles, and at each vertex, the curvilinear triangles with that vertex can be arranged in a cycle such that each curvilinear triangle has an edge at that vertex in common with the next curvilinear triangle (including cyclically the last and the first).

Finally, we can arrange the homeomorphisms from the triangles in $\mathbf{R}^2$ to the curvilinear triangles in the surface so that on edges they result in linear maps from one triangle in $\mathbf{R}^2$ to another. Such a map either reverses or preserves orientation of the edge.

Given such a triangulation, we may find a model for S as a quotient space of a disk (or polygon) with arcs of its boundary (edges) identified in pairs. To do this, pick one curvilinear triangle, call it T_1, and map it to a triangle T_1' in $\mathbf{R}^2$ as above. Now pick a curvilinear triangle, call it T_2, which shares an edge e_1 with T_1, and map it to T_2' in $\mathbf{R}^2$. Note that we can assume that T_i' and T_2' have a common edge corresponding to e_1. (This may require a linear transformation of T_2' which includes a 'flip' in order to get the orientation of the coincident edge right.) Continue this process until we have mapped all the curvilinear triangles onto triangles in a connected polygonal figure P in $\mathbf{R}^2$.

Each external edge e' of the figure P corresponds to an edge e in the surface which is the common of edge of exactly two curvilinear triangles. The other triangle must appear somewhere else in P, and one of its external edges e'' also corresponds to e. Hence, there is a linear map identifying e' with e'' through e. It is clear that the resulting polygonal figure P with pairs of external edges identified is homeomorphic to S.

One problem remains. We must show that P is simply connected, i.e., that it is homeomorphic to a disk. This follows by the following argument. We start with a simply connected polygon T_1'. At each stage we 'glue' a triangle to it on one edge. This process clearly yields a simply connected polygon. (If you have any doubts, you could always use the Seifert-VanKampen Theorem!) Hence, the final result is simply connected. By a suitable mapping, we may assume P is the unit disk with its boundary divided into $2n$ equal arcs equivalent in pairs. Alternately, we may take it to be a regular $2n$-gon with edges identified in pairs.

To complete the proof of the classification theorem, we show how to apply transformations by 'cutting and pasting' to get it in one of the forms discussed in the previous section.

(0) If at any point in the discussion, we are reduced to precisely two 'edges', then S is either a S^2 or $\mathbf{R}P^2$.

Assume now that the number of edges is at least four.

(1) If there are adjacent edges $a\overline{a}$, then they may be collapsed.

We assume this reduction is automatically made wherever possible.

(2) We may assume there is only one vertex under equivalence. For, if not all vertices are equivalent, we can always find an inequivalent pair P, Q at opposite ends of some edge a. The diagram below indicates a cutting and pasting operation which which eliminates one occurence of P replacing it by an occurrence of Q.

(3) If an edge a occurs in two places with the same orientation, we may assume these two ocurrences are adjacent.

(4) If after applying these transformations, we only have adjacent pairs with the same orientation, then the result is homeomorphic to a connected sum of projective planes. Suppose instead that there is a pair $c \ldots \overline{c}$ separated by other edges.

In fact, c and $\overline{c}$ must be split by another such pair $d, \overline{d}$ as indicated above $(\ldots c \ldots d \ldots \overline{c} \ldots \overline{d})$. For otherwise, the vertices on one side of the pair $c, \overline{c}$ could not be equivalent to those on the other side. Note also that the intervening pair $d, \overline{d}$ must also have opposite orientations since we have assumed that all pairs with the same orientation are adjacent. The diagram below indicates how to cut and paste so as to replace the c, d pairs with one of the form $a\overline{a}b\overline{b}$.

To complete the proof, just remember that the connected sum of a 2-torus and a projective plane is homeomorphic to the connected sum of three projective planes. Thus, if there is only one adjacent pair, we may iteratively transform torii into pairs of projective planes.

3. Some Remarks about Higher Dimensional Manifolds

We have shown above that the fundamental group of a compact 2-manifold completely determines the homeomorphism class of the manifold. Also, two surfaces are homeomorphic if and only if they have the same homotopy type.

In general, arbitrary spaces can have the same homotopy type without being homeomorphic, e.g., all contractible spaces have the same homotopy type. Another example would be an open cylinder and an open Moebius band (omitting the boundary in each case). S^1 is a deformation retract of either, but they are not homeomorphic. (Why?)

Thus, non-compact surfaces can have the same homopty type without being homeomorphic.

What about higher dimensional manifolds. It turns out that there are compact three manifolds with the same homotopy type but which are not homeomorphic.

Example 5.6 (Lens Spaces). Let p and q be relatively prime positive integers. Let G_p be the subgroup of $\mathbf{C}^*$ generated by $\zeta = \zeta_p = e^{2\pi i/p}$. G_p is cyclic group of order p. Define an action of G_p on S^3 as follows. Realize S^3 as the subspace $\{(z, w) \in \mathbf{C}^2 \mid |z|^2 + |w|^2 = 1\}$ of $\mathbf{C}^2$ (i.e.,$\mathbf{R}^4$). Let $\zeta^n(z, w) = (\zeta^n z, \zeta^{qn} w)$. The orbit space $L(p, q) = S^3/G_p$ is a compact three manifold with fundamental group $\mathbf{Z}/p\mathbf{Z}$. It is possible to find pairs (p, q) and (p, q') such that $L(p, q)$ and $L(p, q')$ are homotopy equivalent but not homeomorphic. This is quite difficult and we shall not discuss it further.

The classification of compact n-manifolds for $n \geq 3$ is extremely difficult. A first step would be the *Poincaré Conjecture* which asserts that any compact simply connected 3-manifold is homeomorphic to S^3. (It is possible to show that any such 3-manifold has the same homotopy type as S^3.) This is one of the famous as yet unsolved conjectures of mathematics. However, for $n > 3$, the corresponding question, while difficult, has been answered. In 1960, Smale showed that for $n > 4$, any compact n-manifold with the same homotopy type as S^n is homeomorphic to S^n. In 1984, Michael Freedman proved the same result for $n = 4$. (For $n > 3$, being simply connected alone does not imply that a compact n-manifold is homotopy equivalent to S^n. For example, $S^2 \times S^2$ is simply connected. We shall see later in this course that it is not homotopy equivalent to S^4.)

4. An Introduction to Knot Theory

A knot is defined to be a subspace of $\mathbf{R}^3$ which is homeomorphic to S^1.

It is clear intuitively that some knots are basically the same as others, but it is not obvious how to make this precise. (Try making trefoil knot out of string and then manipulate it into a trefoil knot with the opposite orientation!) One idea that comes to mind is to say that two knots are equivalent if one can be continuously deformed into the other. However, since a knot has no thickness, you can 'deform' any knot into a simple unknotted circle by 'pulling it tight' as indicated below.

A sensible definition of equivalence requires some concepts not available to us now, so we shall not go into the matter further. However, some thought suggests that the properties of the *complement* of the knot in $\mathbf{R}^3$ should play a significant role, since it is how the knot is imbedded in $\mathbf{R}^3$ that is crucial. With this in mind, we shall call the fundamental group $\pi(\mathbf{R}^3 - K)$ the knot group of the knot K. It is plausible that any transformation which converts a knot K_1 to an 'equivalent' knot K_2—however that should be defined—will not change the knot group up to isomorphism. (Note that it is not obvious that $\mathbf{R}^3 - K$ is path connected. For example, if K were just an image of S^1 rather than a homeomorph of S^1.)

Any knot K is a compact subset of $\mathbf{R}^3$ so, by an exercise, we know that the inclusion $\mathbf{R}^3 - K \to S^3 - K$ induces an isomorphism of fundamental groups $\pi(\mathbf{R}^3 - K) \cong \pi(S^3 - K)$. It is often more convenient to use the latter group.

The simplest knot—really an 'unknot'—is a circle which we may think of as the unit circle in the x_1, x_2-plane. We shall show that its knot group is $\mathbf{Z}$. To this end, consider K to be imbedded in a closed solid torus A of 'large' radius $\sqrt{2}$ and 'small' radius 1, centered on the x_3-axis. (The reason for this particular choice of A will eventually be clear.)

The boundary of A is a 2-torus T. Let B be the closure in S^3 of the complement of A.

LEMMA 5.7. *B is homeomorphic to a solid torus with T corresponding to its boundary.*

We shall defer the proof of this lemma, but the diagram below hints at why it might be true.

It is clear that B is a deformation retract of $S^3 - K$. (Just project the inside of $A - K$ onto the boundary T of A.) However, since B is a solid torus, its 'center circle' is a deformation retract. It follows that $S^3 - K$ has the same homotopy type as a circle, so its fundamental group is $\mathbf{Z}$.

More interesting are the so-called torus knots. Let $n > m > 1$ be a pair of relatively prime integers. Let $p : \mathbf{R}^2 \to T$ be the the universal covering the torus T. In $\mathbf{R}^2$, consider the the linear path $h : I \to \mathbf{R}^2$ starting at $(0,0)$ and ending at (n,m). (Note that since $(m,n) = 1$, it doesn't pass through a lattice point in between the two ends.) Then $p \circ h$ is a loop in T, and viewing T as imbedded in $\mathbf{R}^3$ in the usual way, the image K of $p \circ h$ is a knot in $\mathbf{R}^3$. K is called a *torus knot* of type (m,n).

Using the $T \simeq S^1 \times S^1$, we may project on either factor.

It is clear from the diagram above, that the projection of $p \circ h$ on the first factor has degree n and on the second factor has degree m. This is a precise way to say that the knot K goes around the torus n times in one direction and m times in the other.

We propose to determine the fundamental group $\pi(S^3 - K)$ for a torus knot and to show that we may determine the integers m, n from it. We shall see that we can recover the type $n > m > 1$ from the knot group. This gives us a way to distinguish one torus knot from another. However, unlike the case of compact surfaces, the fundamental group made abelian $\Pi/[\Pi,\Pi]$ will not suffice. (In fact, it is generally true for a knot group that $\Pi/[\Pi,\Pi] \cong \mathbf{Z}$, so the group made abelian is of no use whatsoever in distinguishing one knots from another. We shall not try to prove this general fact now, although you might try to do it for torus knots.)

Our method will be to use the Seifert-VanKampen Theorem. The analysis above suggests that the decomposition

$$S^3 - K = A - K \cup B - K$$

might work. Unfortunately, the sets on the right are not open. To get around this difficulty, we need to 'fatten' them slightly, but this must be done very carefully to be of any use.

First form a very thin open tube N of radius ϵ about K.

$S^3 - N$ is a deformation retract of $S^3 - K$, so we may work with it instead. Next, let U be the open solid torus obtained by letting A 'grow' by a small amount, say $\epsilon/2$ but not including the bounding torus. Note that $U - N \cap U$ is homeomorphic to an open solid torus. It looks like U with a groove cut in its surface following the knot K. $U - U \cap N$ has as deformation retract its center circle, so $\pi(U - U \cap N) \cong \mathbf{Z}$ with that central circle as generator. The retraction may be accomplished by first including $U - U \cap N$ in U and then retracting to the central circle. We may also imbed T in U and retract to the center circle and in so doing, one of the two generators of $\pi(T)$ retracts to what we may identify as a generator α of $\pi(U - U \cap N)$. Similarly, let V be the open solid torus obtained by letting B encroach inward on A by $\epsilon/2$. As above $V - V \cap N$ is also homeomorphic to a solid torus, and $\pi(V - V \cap N) \cong \mathbf{Z}$. We may identify a generator β of $\pi(V - V \cap N)$ as the retraction of the *other* generator of $\pi(T)$. Now consider the decomposition

$$S^3 - N = (U - U \cap N) \cup (V - V \cap N).$$

The intersection $U \cap V - U \cap V \cap N$ may be viewed as a band going around T.

Its center is a simple loop which is a deformation retract of $U \cap V - U \cap V \cap N$, so

$$\pi(U \cap V - U \cap V \cap N) \cong \mathbf{Z}.$$

Let γ be the generator just described. It is represented by a loop in T with image the knot K shifted over slightly. The commutative diagram

allows us to identify the image of γ in $\pi(U - U \cap N)$. In fact, by the degree argument mentioned above, since it is basically the same as the knot K, it maps either to m times a generator or n times a generator depending how T has been imbedded in $\mathbf{R}^3$. Suppose we choose the imbedding so $\gamma \mapsto \alpha^m$. A similar argument shows that under $\pi(U \cap V \cap -U \cap V \cap N) \to \pi(U - U \cap N, \gamma \mapsto \beta^n$. It follows from the Seifert–VanKampen Theorem that $\pi(S^3 - N)$ is the free product of an infinite cyclic group generatored by α with an infinite cyclic group generated by β modulo the normal subgroup generated by all elements of the form $\alpha^{mi}\beta^{-ni}$. It is not too hard to see from this that $\Pi = \pi(S^3 - K) \cong \pi(S^3 - N)$ is isomorphic to the free group on two generators α, β modulo the relation $\alpha^m = \beta^n$.

We shall now show how we can recover $n > m > 1$ from this group. First consider the subgroup C generated by $\alpha^m = \beta^n$. It is clearly a central subgroup since every element in it commutes with both α and β. The quotient group $\overline{\Pi}$ is the free product of a cyclic group of order m generated by $\overline{\alpha}$ and a cyclic group of order n generated by $\overline{\beta}$. (Just check that Π/C has the appropriate universal mapping property.) However, it is generally true that any free product has trivial center. (Exercise.) From this it follows that C is the center of Π. We can recover m, n from Π as follows. First, since $\overline{\Pi} = \Pi/C \cong \mathbf{Z}/m\mathbf{Z} * \mathbf{Z}/n\mathbf{Z}$, it follows that the group made abelian $\overline{\Pi}/[\overline{\Pi}, \overline{\Pi}] \cong \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$. (Again, just use a univeral mapping property argument.) Hence, its order is mn. On the other hand, it is true that any element of finite order in $\mathbf{Z}/m\mathbf{Z} * \mathbf{Z}/n\mathbf{Z}$ must be conjugate to an element in $\mathbf{Z}/m\mathbf{Z}$ or to an element in $\mathbf{Z}/n\mathbf{Z}$. (Exercise.) Hence, the maximal order of any element of finite order in $\overline{\Pi}$ is n (since $n > m > 1$). Thus $\overline{\Pi} \cong \Pi/C$—which is Π modulo its center and so does not depend on any particular presentation of Π—contains enough information for us to recover n and mn, and hence m.

It follows that if K_1 and K_2 are torus knots of types $n_1 > m_1 > 1$ and $n_2 > m_2 > 1$ respectively, then they have the same knot groups if and only if $n_1 = n_2$ and $m_1 = m_2$.

To complete the discussion, we prove the Lemma which asserts that $B = S^2 - A^\circ$ is homeomorphic to a solid torus. To this end we describe stereographic projection from $(0,0,0,1)$ on S^3 to $\mathbf{R}^3$ imbedded as the hyperplane $x_4 = 0$. Let (x_1, x_2, x_3, x_4) be a point on S^3 $({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 1)$ and let $(y_1, y_2, y_3, 0)$ be the corresponding point in $\mathbf{R}^3$. Then

$$\begin{aligned} x_1 &= 0 + t(y_1 - 0) = ty_1 \\ x_2 &= 0 + t(y_2 - 0) = ty_2 \\ x_3 &= 0 + t(y_3 - 0) = ty_3 \\ x_4 &= 1 + t(0 - t) = 1 - t. \end{aligned}$$

Put this in the equation of S^3 to obtain

$$1 = t^2({y_1}^2 + {y_2}^2 + {y_3}^2) + t^2 - 2t + 1.$$

Put $\rho = \sqrt{{y_1}^2 + {y_2}^2 + {y_3}^2}$. Then we get

$$t = \frac{2}{\rho^2 + 1}.$$

Thus

$$\begin{aligned} x_1 &= \frac{2y_1}{\rho^2 + 1} \\ x_2 &= \frac{2y_2}{\rho^2 + 1} \\ x_3 &= \frac{2y_3}{\rho^2 + 1} \\ x_4 &= \frac{\rho^2 - 1}{\rho^2 + 1}. \end{aligned}$$

Now consider the surface obtained by intersecting ${x_1}^2 + {x_2}^2 = u^2$ with S^3. This may also be described by ${x_3}^2 + {x_4}^2 = v^2$ where $u^2 + v^2 = 1$. We may assume here that $0 \le u, v \le 1$. This surface maps under stereographic projection to

$$u^2(\rho^2 + 1)^2 = 4({y_2}^2 + {y_2}^2).$$

Put $r = \sqrt{{y_1}^2 + {y_2}^2}$. Doing some algebra yields the equation

$$(r - \frac{1}{u})^2 + {y_3}^2 = \left(\frac{v}{u}\right)^2.$$

This is a 2-torus in $\mathbf{R}^3$ with 'large' radius $\dfrac{1}{u}$ and 'small' radius $\dfrac{v}{u}$. $u = 0, v = 1$ is a special case. That means $x_1 = x_2 = 0$ and the locus on S^3 is a circle rather than a surface. The stereographic projection of this circle less the point $(0,0,0,1)$ is the y_3-axis. Similarly, $v=0, u=1$ is a special case with the stereographic projection being the circle $y_3=0, r=1$. The torii for $\frac{1}{\sqrt{2}} \leq u \leq 1$ fill out the solid torus A. However, the set B (which is the closure of the complement of A in S^3) may be described by $0 \leq u \leq \frac{1}{\sqrt{2}}$ or equivalently $\frac{1}{\sqrt{2}} \leq v \leq 1$. However, interchanging x_1 with x_3 and x_2 with x_4 clearly provides a homeomorphism of A with B. It follows that B is also a solid torus and the common boundary of A and B boundary is given by $u=v=\frac{1}{\sqrt{2}}$. This corresponds to the torus of 'large' radius $\sqrt{2}$ and small radius 1, i.e., to T. Note that the generator of $\pi(B)$ corresponds to the 'other' generator of $\pi(T)$ as required.

CHAPTER 6

Singular Homology

1. Homology, Introduction

In the beginning, we suggested the idea of attaching algebraic objects to topological spaces in order to discern their properties. In language introduced later, we want functors from the category of topological spaces (or perhaps some related category) and continuous maps (or perhaps homotopy classes of continuous maps) to the category of groups. One such functor is the fundamental group of a path connected space. We saw how to use such functors in proving things like the Brouwer Fixed Point Theorem and related theorems.

The fundamental group is the first of a sequence of functors called homotopy groups. These are defined roughly as follows. Let X be a space and fix a base point x_0. For each $n \geq 1$ consider the set $\pi_n(X, x_0)$ of base point preserving homotopy classes of maps of S^n into X. Note that $\pi_1(X, x_0)$ is just the fundamental group. It is possible with some care to defined a group structure on $\pi_n(X, x_0)$. (Think about the case $n = 1$ and how you might generalize this to $n > 1$.) The resulting group is abelian for $n > 1$ and is called the nth homotopy group. It is also not too hard to see that a (base point preserving) map of spaces $X \to Y$ induces a homomorphism of corresponding homotopy groups.

The homotopy groups capture quite a lot about the geometry of spaces and are still the subject of intense study. Unfortunately, they are very difficult to compute. However, there is an alternate approach, the so-called homology groups which historically came first. The intuition behind homology groups is a bit less clear, but they are much easier to calculate than homotopy groups, and their use allows us to solve many important geometric problems. Also, homology theory is a basic tool in further study of the subject. We shall spend the rest of this year studying homology theory and related concepts. You will get to homotopy theory later if you continue with your study of algebraic topology.

One way to explain to roots of homology theory is to consider the basic integral theorems of vector analysis. (You probably studied this as an undergraduate, and in any case you will probably have to teach it as a teaching assistant.) Green's Theorem asserts that for an appropriate region D in $\mathbf{R}^2$ and a differential form $Pdx + Qdy$, we have

$$\int_{\partial D} Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

Note that the orientation of the boundary plays an important role.

We want to know the theorem for fairly arbitrary regions, including those with 'holes' where the boundary may be disconnected. The strategy for proving it is first to do it for regions with a relatively simple shape, e.g., 'curvilinear' triangles, and then approach general regions by 'triangulating' them as discussed previously in the section on classifying surfaces.

Here two important concepts play a role. First, we have the concept of a 'chain' as a formal sum of 'triangles' (or perhaps other elementary regions). Such an object helps us think about dissecting the region for integration on the right of the formula. Secondly, we see that the *oriented* boundary of such a chain will break up into many separate segments, some of which will be the same segment repeated twice but with opposite orientation. In the integral on the right, such segments will cancel and we will be left only with the integral on the external (oriented) boundary. A useful way to think of this is that we have an algebraic boundary which attaches to any chain the formal sum of its boundary segments but where segments in opposite directions are give opposite signs so they cancel in the formal sum.

Let's focus on the external boundary ∂D of the region. This can be viewed as a formal sum of segments, and each segment has a boundary consisting of its two end points with opposite 'orientation' or sign. Since the boundary is closed, if we take its algebraic boundary in this sense, each division point is counted twice with opposite signs and so the zero dimensional 'boundary' of ∂D is zero.

Note however, that other closed loops may have this property. e.g., the unit circle in $\mathbf{R}^2 - \{(0,0)\}$. A closed curve (or collection of such) decomposed this way into segments is called a *cycle*. The extent to which cycles in a region X differ from boundaries of subregions D is a question which is clearly related to the fundamental group of the region. In fact, we shall measure this property by a functor called the first homology group $H_1(X)$. It will turn out that $H_1(X)$ is $\pi_1(X)$ made abelian!

The above considerations can be generalized to regions in $\mathbf{R}^n$ for $n > 2$. In $\mathbf{R}^3$, there is a direct generalization of Green's Theorem called Stokes's Theorem. The relates a line integral over the boundary of a surface to a surface integral over the surface. (If you don't remember Stokes's Theorem, you should go now and look it up.) The same considerations apply except now the curvilinear triangles dissect the surface. Another generalization is $Gauss's$ theorem which asserts for a solid region D in $\mathbf{R}^3$

$$\iint_{\partial D} P dy\, dz + Q dx\, dz + R dx\, dy = \iiint_D \left(\frac{\partial P}{\partial X} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV.$$

The first integral is a surface integral of a 'two-form' and the second is an ordinary triple integral. The orientation of the surface relative the the region D plays an crucial role. We have stated the theorem the way it was commonly stated in the 19th century. You may be more familiar with the more common form using vector fields. The proof of this theorem parallels that of Green's theorem. We first prove it for basic regions such as 'curvilinear tetrahedra'. We then view D as dissected as an appropriate union of such tetrahedra. This leads to the concept of a 3-chain as a formal sum of tetrahedra. As above, the boundary of a tetrahedron can be thought of as a formal sum of 'triangles' with appropriate orientations. The same triangle may appear as a face on adjacent tetrahedral cells with opposite orientations. In the sum of all the surface integrals for these cells, these two surface integrals cancel, so we are left with the integral over the external boundary. This cancellation can be treated formally in terms of algebraic cancellation of the terms in the formal sums.

Let X be a solid region in $\mathbf{R}^3$. As above, the boundary of a subregion D dissected into triangles may be though of as a 2-cycle (because its boundary is trivial), but there may be other closed surfaces in X which don't bound a subregion. Thus again we have the question of the extent to which 2-cycles differ from boundaries. This is measured by a group called the second homology group and denoted $H_2(X)$.

The issue of when cycles are boundaries is clearly something of some significance for understanding the geometry of a region. It is also important for other reasons. A differential form $Pdx + Qdy$ defined on a path connected region D is called exact if it is of the form df for some function f defined on D. It is important to know if a differential form is exact when we want to solve the differential equation $Pdx + Qdy = 0$. (Why?) It is easy to see that every exact form satisfies the relation $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$. Such forms are called closed. We would like to know if every closed form on D is exact. One way to approach this issue is as follows. Choose an arbitrary point x_0 in D and given any other point x, choose a path $\mathcal{C}$ from x_0 to x and define $f(x) = \int_{\mathcal{C}} Pdx + Qdy$. If this function f is well defined, it is fairly obvious that $df = Pdx+Qdy$. Unfortunately, f is not always well defined because the integral on the right might depend on the path. Consider two different paths $\mathcal{C}$ and $\mathcal{C}'$ going from x_1 to x.

For simplicitly assume the two paths together bound a subregion of D. It is easy to see by an application of Green's theorem that the two line integrals are the same. Hence, if every cycle is a boundary, it follows that the function is well defined. Hence, in that case, every closed form is exact.

This sort of analysis is quite fascinating and lies at the basis of many beautiful theorems in analysis and geometry. We hope you will pursue such matters in other courses. Now, we shall drop our discussion of the motivation for homology theory and begin its formal development

2. Singular Homology

The material introduced earlier on abelian groups will be specially important, so you should review it now. In particular, make sure you are comfortable with free abelian groups, bases for such groups, and defining homomorphisms by specifying them on bases.

The *standard n-simplex* is defined to be the set Δ^n consisting of all $(t_0, t_1, \dots, t_n) \in \mathbf{R}^{n+1}$ such that $\sum_i t_i = 1$ and $t_i \geq 0, i = 0, 1, \dots n$. (Note that we start the numbering of the coordinates with 0.) The cases $n = 0, 1, 2$ are sketched below. Δ^3 is a solid tetrathedron but imbedded in $\mathbf{R}^4$.

We could of course choose the standard simplex in $\mathbf{R}^n$, but imbedding it in $\mathbf{R}^{n+1}$ gives a more symmetric description and has some technical advantages.

Let X be a space. A *singular n-simplex* is any (continuous) map $\sigma : \Delta^n \to X$. You can think of this as the generalization of a 'curvilinear triangle', but notice that it need not be one-to-one. For example, any constant map is a singular n-simplex. Note that a 0-singular simplex can be identified with a point in X. A singular n-chain is any formal linear combination $\sum_i n_i\sigma_i$ of singular n-simplices with integer coefficients n_i. (Note, by implication, 'linear combination' always means 'finite linear combination'. Unless explicitly stated, in any sum which potentially involves infinitely many terms, we shall always assume that *all but a finite number of terms are zero*.)

More explicitly, let $S_n(X)$ be the free abelian group with basis the set of all singular n-simplices in X. Note that this is an enormous group since the basis certainly won't be countable for any interesting space. Thus, $S_0(X)$ can be viewed as the free abelian group with X as basis. A *singular n-chain* then is any element of this group. By the usual conventions, $S_n(X)$ is the trivial group $\{0\}$ for $n < 0$ since it is the free abelian group on the empty basis.

A bit of explanation for this definition is called for. Suppose X is a compact surface with a triangulation. Each 'triangle' may be viewed as a singular 2-simplex where the map is in fact one-to-one. Also, there are restrictions about how these triangles intersect. In fact, up to homeomorphism, we may view X as a two dimensional 'polyhedron' (but we may not be able to imbed it in $\mathbf{R}^3$ if it is not orientable). The generalization of this to higher dimensions is called a *simplicial complex*. Simplicial complexes and their homeomorphs were the original subject matter of algebraic topology. The reason is fairly clear. As in the previous section, there are many reasons to consider formal sums of the triangles (in general n-simplices) in a triangulation. Unfortunately, we are then left with the problem of showing that the results we obtain are not dependent on the triangulation. This was the original approach. Our approach will allow us to avoid this problem. We consider arbitrary singular n-simplices and linear combinations of such. This theory will introduce many *degenerate* n-simplices, but there is no question that it depends only on the space X. This large advantage of singular theory is counterbalanced by the fact that it is difficult to make computations because the groups invovled are so large. Hence, later we shall introduce simplicial complexes and related objects where the groups have much smaller, even finite, bases in most cases, so the computations are much easier.

The considerations in the previous section suggest that we want to be able to define a homomorphism $\partial_n : S_n(X) \to S_{n-1}(X)$ which reflects algebraically the properties of an oriented boundary. Since $S_n(X)$ is free on the set of singular n-simplices $\sigma : \Delta^n \to X$, it suffices to define ∂_n on all singular n-simplices σ and then extend by linearity. A moment's thought suggests that the boundary should be determined somehow by restricting σ to the 'faces' of the standard simplex and then summing with appropriate signs to handle the issue of orientation. In order to get this right, we need a rather long digression on so called affine simplices.

Let $T = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\}$ be any finite set of points in $\mathbf{R}^{n+1}$. We shall call this set *affinely independent* if the set of differences $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_p - \mathbf{x}_0\}$ is linearly independent. (You should prove this does not depend on the use of $\mathbf{x}_0$ rather than some other element in the set.) We shall assume below that any such set is affinely independent, but some of what we do will in fact work for any finite set. The set $\{\sum_i t_i \mathbf{x}_i \,|, t_i \in \mathbf{R}, \sum_i t_i = 1, t_i \geq 0\}$ is called the *affine simplex* spanned by $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\}$. It is not too hard to see that it is a closed convex set, and in fact it is the smallest convex set containing the set T. We denote it $[T] = [\mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_p]$.

The set A of all points of the form $\sum_{i=0}^p t_i \mathbf{x}_i$ where $t_i \in \mathbf{R}$ and $\sum_i t_i = 1$ is called the affine subspace spanned by those points. It is a translate (by $\mathbf{x}_0$ for example) of the linear subspace of $\mathbf{R}^{n+1}$ spanned by $\{\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_p - \mathbf{x}_0\}$. Namely,

$$\sum_{i=0}^p t_i \mathbf{x}_i = \sum_{i=0}^p t_i(\mathbf{x}_i - \mathbf{x}_0) + (\sum_{i=0}^p t_i)\mathbf{x}_0 = \sum_{i=1}^p t_i(\mathbf{x}_i - \mathbf{x}_0) + \mathbf{x}_0.$$

It follows from this that if two points are in A, then the line they determine is in A. Also, (in the default case where the points are independent), this set is homeomorphic to $\mathbf{R}^p$, and we call it a p-dimensional affine subspace. (Since $[\mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_p]$ is the closure of an open subset of this space, it is also reasonable to consider it a p-dimensional object.) (Abstractly, an affine subspace A may be defined as any subspace having the property $\mathbf{x}, \mathbf{y} \in A, s, t \in \mathbf{R}, s + t = 1 \to s\mathbf{x} + t\mathbf{y} \in A$. The p dimensional affine subspaces of $\mathbf{R}^{n+1}$ are exactly the same as the translates of p-dimensional linear subspace.)

The coefficients $(t_0, t_1, \dots, t_p)$ of a point $\sum_i t_i \mathbf{x}_i$ in the affine subspace are determined uniquely by the point. We leave this as an exercise for the student. These coefficients are called the *barycentric coordinates* of the point. The reason for the terminology is as follows. If you imagine weights $m_0, m_1, \dots, m_p$ at the points $\mathbf{x}_0, \dots, \mathbf{x}_p$, then the center of mass of these points is $\dfrac{1}{\sum_i m_i} \sum_i m_i \mathbf{x}_i$. Setting $t_i = \dfrac{m_i}{\sum_i m_i}$, we get the barycentric coordinates of the center of mass. This makes a lot of sense if the masses are all non-negative (as it the case for points in the simplex $[\mathbf{x}_0, \dots, \mathbf{x}_p]$) but we may also allow negative 'masses' (e.g., charges) and thereby encompass arbitrary points in the affine space as weighted sums.

For linear spaces, the nicest maps are linear maps, and there is a corresponding concept for affine spaces. Let $f : A \to B$ be a function which maps an affine subspace of $\mathbf{R}^{n+1}$ to an affine subspace of $\mathbf{R}^{m+1}$. n and m may be different as may be the dimensions of A and B. We shall say f is an affine map if for any points $\mathbf{x}, \mathbf{y} \in A$, we have $f(t\mathbf{x} + s\mathbf{y}) = tf(\mathbf{x}) + sf(\mathbf{y})$ for all $t, s \in \mathbf{R}$ such that $t + s = 1$. Note that this implies that f carries any line in A into a line in B or a point in B. It is not hard to show that an affine map is completely determined by its values on an affinely independent generating set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p\}$ through the formula $f(\sum_i t_i \mathbf{x}_i) = \sum_i t_i f(\mathbf{x}_i)$ where $\sum_i t_i = 1$, and moreover an affine map may be defined by specifying it arbitrarily on such a set. Clearly, if the image points are also independent, then $f([\mathbf{x}_0, \dots, \mathbf{x}_p]) = [f(\mathbf{x}_0), \dots, f(\mathbf{x}_p)]$.

Note that any p-dimensional affine simplex $[\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_p]$ in $\mathbf{R}^{n+1}$ is a singular simplex under the affine map $\Delta^p \to \mathbf{R}^{n+1}$ defined by $\mathbf{e}_i \mapsto \mathbf{x}_i$. We shall abuse notation by using the symbol $[\mathbf{x}_0, \dots, \mathbf{x}_p]$ both for this singular simplex (which is a map) and for its image.

Consider now the standard simplex Δ^n. Let $\mathbf{e}_i = \mathbf{e}_i^{n+1}$ denote the ith standard basis vector in $\mathbf{R}^{n+1}$. Clearly, $\Delta^n = [\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n]$. (Under our abuse of notation, this means Δ^n also denotes the identity map of the standard simplex.) The barycentric coordinates of Δ^n

are just the ordinary coordinates in $\mathbf{R}^{n+1}$. The affine $(n-1)$-simplex $[\mathbf{e}_0, \dots, \hat{\mathbf{e}}_i, \dots, \mathbf{e}_n]$ is called the ith face of Δ^n. (Here we adopt the convention that putting a 'hat' over an element of a list means that it should be omitted from the list.) It is in effect obtained by intersecting Δ^n with the hyperlane with equation $t_i = 0$. With our abuse of notation, these faces are the images of the singular simplices

$$\epsilon_i^n = [\mathbf{e}_0, \dots, \hat{\mathbf{e}}_i, \dots, \mathbf{e}_n] : \Delta^{n-1} \to \Delta^n$$

(This is a shorthand way of saying that

$$\begin{aligned} \epsilon_i^n(\mathbf{e}_j^{n-1}) &= \mathbf{e}_j^n \qquad j < i \\ &= \mathbf{e}_{j+1}^n \qquad j \geq i.) \end{aligned}$$

These maps are called the face maps for Δ^n. Note that there are $n+1$ faces of an n-simplex.

We are now in a position to define the map $\partial_n : S_n(X) \to S_{n-1}(X)$. Let $\partial_n = 0$ for $n \leq 0$. For $n > 0$, for a singular n-simplex $\sigma : \Delta^n \to X$, define

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^n.$$

Note that each of the terms on the right is a singular $(n-1)$-simplex which is in some sense the restriction of σ to the ith face of σ. The diagram below shows why in the cases $n = 1, 2$ the signs make sense by reflecting a proper orientation for each simplex in the boundary.

PROPOSITION 6.1. $\partial_n \circ \partial_{n+1} = 0$.

This proposition plays an absolutely essential role in all that follows. For, given it, we may now make the following definitions. The subgroup $Z_n(X) = \operatorname{Ker} \partial_n$ of $S_n(X)$ is called the subgroup of n-cycles. Its elements may be thought of as 'closed' n-dimensional objects in X. The subgroup $B_n(X) = \operatorname{Im} \partial_{n+1}$ is called the subgroup of n-boundaries. Because of the lemma, every n-boundary is an n-cycle, i.e., $B_n(X) \subseteq Z_n(X)$. Finally, we shall define the nth homology group of X as the factor group

$$H_n(X) = Z_n(X)/B_n(X).$$

This measures the extent to which it is not true that every n-cycle is an n-boundary.

Proof of Proposition 6.1. Note first that the proof is clear for $n \leq 0$.

For $n > 0$, let $\sigma : \Delta^{n+1} \to X$ be a singular $(n+1)$-simplex. Then

$$\partial_{n+1}\sigma = \sum_{j=0}^{n+1} (-1)^j \sigma \circ \epsilon_j^{n+1}$$

so

$$\begin{aligned}
\partial_n(\partial_{n+1}\sigma) &= \sum_{j=0}^{n+1} (-1)^j \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_j^{n+1} \circ \epsilon_i^n \\
&= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_i^n
\end{aligned}$$

The strategy is to pair terms with opposite terms which cancel. The diagram below illustrates the argument for $n = 1$.

The general case follows from

Lemma 6.2. *For $0 \leq i < j \leq n+1$, $\epsilon_j^{n+1} \circ \epsilon_i^n = \epsilon_i^{n+1} \circ \epsilon_{j-1}^n$.*

This suffices to prove the proposition because the term

$$(-1)^{i+j} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_i^n \qquad 0 \leq i < j \leq n+1$$

cancels the term

$$(-1)^{j-1+i} \sigma \circ \epsilon_i^{n+1} \circ \epsilon_{j-1}^n.$$

The student should check that every term is accounted for in this way. $\square$

Proof of Lemma 6.2. For $0 \leq i < j \leq n+1$, consider the $(n-1)$-dimensional affine simplex $[\mathbf{e}_0^{n+2}, \dots, \hat{\mathbf{e}}_i^{n+2}, \dots, \hat{\mathbf{e}}_j^{n+2}, \dots, \mathbf{e}_{n+1}^{n+2}]$ in Δ^{n+1}. This may be thought of as its intersection with the linear subspace defined by setting $t_i = t_j = 0$. Call this its (i, j)-face. (This is also the image of the affine map which maps the basis elements as below in the indicated order

$$\{\mathbf{e}_0^n, \dots, \mathbf{e}_{n-1}^n\} \to \{\mathbf{e}_0^{n+2}, \dots, \hat{\mathbf{e}}_i^{n+2}, \dots, \hat{\mathbf{e}}_j^{n+2} \dots \mathbf{e}_{n+1}^{n+2}\}.$$

This exhibits each of the $(n+2)(n+1)/2$ (i,j)-faces as a singular $(n-1)$-simplex.)

The right hand side of the equation can be viewed as folows. First take the ith face of Δ^{n+1} by setting $t_i = 0$. The barycentric coordinates of a point on this face relative to its vertices in the proper order (obtained from Δ^n) will be $(s_0, \dots, s_n)$, where $s_r = t_r$ for $r < i$ and $s_r = t_{r+1}$ for $r \geq i$. If we now take its $(j-1)$st face by setting $s_{j-1} = 0$, that is the same thing as setting $t_j = 0$. (Remember $i \leq j-1$.) This gives us the (i,j)-face. The left hand side of the equation also represents the (i,j) face by a similar analysis. First, take the jth face of Δ^{n+1} by setting $t_j = 0$. Then take the ith face of that by setting $t_i = 0$. (Since $i < j$, we do not have to worry about the shift for $r \geq j$.)

□

3. Properties of Singular Homology

The singular homology groups $H_n(X)$ are generally so hard to compute that we won't be able to do much more than simple examples to start. We shall derive a series of results or 'axioms' and then show how we can compute singular homology just by use of these 'axioms'. The point of this is that there are several other versions of homology theory. If we can show that an alternate theory satisfies these same 'axioms', then it will follow that it is essentially the same as singular homology. (This axiomatic approach is due to Eilenberg and Steenrod and is worked out in detail in their book *Foundations of Algebraic Topology*.)

Theorem 6.3 (The Dimension Axiom). *For a space $X = \{x\}$ consisting of a single point, we have*

$$\begin{aligned} H_n(\{x\}) &= \mathbf{Z} \qquad n = 0 \\ &= 0 \qquad n > 0. \end{aligned}$$

Proof. For each $n \geq 0$ there is precisely one singular n-simplex σ_n which is the constant map with value x. For $n > 1$, we have

$$\begin{aligned} \partial_n \sigma_n &= 0 \qquad n \quad \text{odd} \\ \partial_n \sigma_n &= \sigma_{n-1} \qquad n \quad \text{even}. \end{aligned}$$

The reason is that when n is odd there are an even number of terms and they all cancel. In the even case there is one term left over with sign $(-1)^n = 1$. The above situation can be summarized by the diagram

$$\dots \xrightarrow{0} S_2(\{x\}) \xrightarrow{\cong} S_1(\{x\}) \xrightarrow{0} S_0(\{x\}).$$

Then for $n > 0$, $Z_n(\{x\})$ is alternately $\mathbf{Z}\sigma_n$ or zero and $B_n(\{x\})$ is alternately zero or $\mathbf{Z}\sigma_n$. Hence, the quotients are all zero. It is clear that $Z_0(\{x\}) = \mathbf{Z}\sigma_0$ and $B_0(\{x\}) = 0$, so $H_0(\{x\}) = \mathbf{Z}$. □

3.1. Homological Algebra. Algebraic topology generates algebra which is of interest in its own right and in fact has been used extensively outside algebraic topology. We shall periodically introduce such concepts as they are needed. In principle, you could separate these out and have a short course in homological algebra, but all the motivation of course comes from the geometric ideas.

A *chain complex* C consists of a collection of abelian groups C_n, for $n \in \mathbf{Z}$, and maps $d_n : C_n \to C_{n-1}$ with the property that $d_n \circ d_{n+1} = 0$ for each n. (Sometimes the collection of all d_n is denoted 'd' and we say simply $d \circ d = 0$.) Thus the collection of singular n-chains $S_n(X)$ for all n forms a chain complex $S_*(X)$.

Most of the chain complexes we shall consider will be non-negative, i.e., $C_n = 0$ for $n < 0$. *Unless otherwise stated, you should assume the term 'chain complex' is synonymous iwth 'non-negative chain complex'.*

Given a chain complex C, we may define its homology groups as follows. $Z_n(C) = \operatorname{Ker} d_n, B_n(C) = \operatorname{Im} d_{n+1}$ so $B_n(C) \subseteq Z_n(C)$. These are called the cylces and boundaries of the chain complex. Two chains are called *homologous* if they differ by a boundary. We define $H_n(C) = Z_n(C)/B_n(C)$, and its elements are called *homology classes*.

Note that $H_n(S_*(X)) = H_n(X)$.

Chain complexes are the objects of a category. Let C and C' be chain complexes. A morphism $f : C \to C'$ is a collection of group homomophisms $f_n : C_n \to C'_n$ such that $d'_n \circ f_n = f_{n-1} \circ d_n$ for each n.

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ C'_n & \longrightarrow & C'_{n-1} \end{array}$$

It is easy to see that the composition of two morphisms of chain complexes is again such a morphism.

A morphism of $f : C \to C'$ of chain complexes induces a homomorphism $H_n(f) : H_n(C) \to H_n(C')$ for each n. Namely, because f commutes with d, it follows that $f_n(Z_n(C)) \subseteq Z_n(C')$ and $f_n(B_n(C)) \subseteq B_n(C')$ whence it induces a homomorphism on the quotients $H_n(f) = H_n(C) = Z_n(C)/B_n(C) \to H_n(C') = Z_n(C')/B_n(C')$. $(H_n(f)(\bar{c}) = \overline{f_n(c)}.)$

PROPOSITION 6.4. *For each n, the homology group $H_n(\quad)$ is a functor from the category of chain complexes to the category of abelian groups.*

PROOF. It is easy to see that $H_n(\mathrm{Id}) = \mathrm{Id}$, so all we need to do is show is $H_n(g \circ f) = H_n(g) \circ H_n(f)$ for

$$C \xrightarrow{f} C' \xrightarrow{g} C''$$

morphisms of chain complexes. We leave this to the reader to verify. □

We may now apply the above homological algebra to singular homology. Let $f : X \to Y$ be a map of spaces. Define a morphism $f_\sharp : S(X) \to S(Y)$ as follows. For σ a singular n-simplex, let $f_n(\sigma) = f \circ \sigma : \Delta^n \to Y$. Since the set of singular n-simplices forms a basis for $S_n(X)$, this defines a homomorphism $f_n : S_n(X) \to S_n(Y)$. It commutes with the boundary operators since

$$\begin{aligned}\partial_n^Y(f_n(\sigma)) &= \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \epsilon_i^n \\ &= \sum_{i=0}^{n}(-1)^i f_{n-1}(\sigma \circ \epsilon_i^n) \\ &= f_{n-1}(\sum_{i=0}^{n}(-1)^i \sigma \circ \epsilon_i^n) = f_{n-1}(\partial_n^X \sigma).\end{aligned}$$

(Make sure you understand the reasons for each of the steps!)

It follows that $f_\sharp$ induces a homomoprhism

$$H_n(f) = H_n(f_\sharp) : H_n(X) = H_n(S_*(X)) \to H_n(Y) = H_n(S_*(Y)).$$

This homomorphism is often abbreviated f_n and the collection of all of them is often denoted f_*. (This yields a slight notational problem, since we are also using f_n to denote the induced maps of singular n-chains, but usually the context will make clear what is intended. Where there is any doubt, we may use the notation $H_n(f)$ or f_* for the collection of all the maps.)

PROPOSITION 6.5 (Functoriality Axiom). *$H_n(-)$ is a functor from the category of topological spaces to the category of abelian groups*

PROOF. First note that the associations $X \mapsto S_*(X)$ and $f \mapsto f_\sharp$ provide a functor from the category of topological spaces to the category of chain complexes. For, it is clear that the identity goes to the identity and it is easy to verify that $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. It is not hard to see that the composition of two functors is a functor, and since $H_n(-)$ is clearly such a composition, it follows that it is a functor. $\square$

PROPOSITION 6.6 (Direct Sum Axiom). *Let X be the disjoint union $\cup_{a\in A} X_a$ where each X_a is a path connected subspace. Then for each $n \geq 0$, we have $H_n(X) \cong \oplus_{a\in A} H_n(X_a)$.*

If X is itself path connected, then $H_0(X) \cong \mathbf{Z}$. Hence, in the general case $H_0(X)$ is a free abelian group with basis the path components of X.

Note that the indexing set A could be infinite. Also, you should reivew what you know about possibly infinite direct sums of abelians groups.

PROOF. If σ is a singular n-simplex, then the image of σ must be contained in one of the X_a. Since, $S_n(X)$ is free with the set of singular n-simplices as basis, it follows that we can make an identification $S_n(X) = \oplus_{a\in A} S_n(X_a)$. Also, the faces of any singular n-simplex σ with image in X_a will be singular $(n-1)$-simplices with images in X_a. Thus, $\partial_n\sigma$ can be identified with a chain in $S_n(X_a)$. In other words, the above direct sum decomposition is consistent with the boundary operators. It follows that

$$\begin{aligned} Z_n(X) &= \oplus_a Z_n(X_a) \\ B_n(X) &= \oplus_a B_n(X_a) \\ H_n(X) &= \oplus_a H_n(X_a) \end{aligned}$$

Suppose next that X is path connected. $Z_0(X) = S_0(X)$ is free on the constant maps $\sigma : \Delta^0 \to X$ which may be identified with the points $x \in X$. However, given two points, $x_1, x_2 \in X$, there is a path $\sigma : \Delta^1 \to X$ from x_1 to x_2.

It follows that $\partial_1\sigma = x_2 - x_1$ whence any two cycles in $Z_0(X)$ differ by a boundary. Fix any $x_0 \in X$. It follows that $H_0(X)$ is generated by the coset of x_0. We leave it to the student to show that this coset is of infinite order in $H_0(X) = Z_0(X)/B_0(X)$. $\square$

It should be noted that in the above proof, the direct sum decomposition is related to the geometry as follows. Let $i_a : X_a \to X$ be the inclusion of X_a in X. This induces $i_{a,n} : H_n(X_a) \to H_n(X)$ for each n.

The collection of these homomophisms induce

$$\oplus_a i_{a,n} : \oplus_a H_n(X_a) \to H_n(X).$$

Namely, any element of the left hand side can be expressed $(h_a)_{a\in A}$ where $h_a \in H_n(X_a)$ and $h_a = 0$ for all but a finite number of a. The map is

$$(h_a)_{a\in A} \mapsto \sum_{a\in A} i_{a,n}(h_a).$$

This homomorphism is in fact one of the two inverse isomorphisms of the Proposition.

3.2. Homotopies. We want to show that homotopic maps $f, g : X \to Y$ induce the same homomorphisms $f_n = g_n : H_n(X) \to H_n(Y)$ of singular homology groups. This requires introducing a certain kind of algebraic construction called a chain homotopy which arises out of the geometry.

To see how this comes about, let $F : X \times I \to Y$ be a homotopy from f to g, i.e., $F(x, 0) = f(x), F(x, 1) = g(x)$ for $x \in X$. Let $\sigma : \Delta^n \to X$ be a singular n-simplex in X. This induces a map $\sigma \times \mathrm{Id} : \Delta^n \times I \to X \times I$ which is called appropriately a *singular* $(n + 1)$*-prism* in $X \times I$. F on the 'bottom' of this prism is basically $f_n(\sigma)$ and on the 'top' is $g_n(\sigma)$. The top and bottom are part of the boundary (in the naive sense) of the prism. The rest of the boundary might be described loosely as '$\partial_n\sigma \times I$' which is a 'sum' of prisms of one lower dimension based on the faces of σ.

The basic algebra we need reflects the geometry of the prism. We shall describe it for the standard prism $\Delta^n \times I$ where we can make do with affine maps. Our first difficulty is that a prism is not a simplex, so we have to subdivide it into simplices. We view $\Delta^n \times I$ as a subset of $\mathbf{R}^{n+2}$ with standard basis $\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$. Then Δ^n may be viewed as the affine simplex spanned by the first $n + 1$ basis vectors (the bottom of the prism), i.e., by setting $t = 0$. Let $\overline{\Delta}_n$ denote the affine simplex spanned by $\overline{\mathbf{e}}_0 = \mathbf{e}_0 + \mathbf{e}_{n+1}, \dots, \overline{\mathbf{e}}_n = \mathbf{e}_n + \mathbf{e}_{n+1}$ (the top of the prism), i.e., by setting $t = 1$. Define

$$p_n = \sum_{i=0}^{n} (-1)^i [\mathbf{e}_0, \dots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \dots, \overline{\mathbf{e}}_n].$$

As a sum of affine simplices in the space $\Delta_n \times I$, p_n may be viewed as an $(n+1)$-chain in $S_{n+1}(\Delta^n \times I)$. It corresponds to a decomposition of the prism into simplices with signs to account for orientations. The diagrams below show pictures for $n = 1, 2$.

More generally, if $\alpha = [\mathbf{x}_0, \dots, \mathbf{x}_n]$ is any affine n-simplex (in some Euclidean space), define

$$p_n(\alpha) = \sum_{i=0}^{n} (-1)^i [\mathbf{x}_0, \dots, \mathbf{x}_i, \overline{\mathbf{x}}_i, \dots, \overline{\mathbf{x}}_n].$$

Viewing $\alpha : \Delta^n \to \operatorname{Im} \alpha$ as a map, we have have a corresponding map of prisms $\alpha \times \mathrm{Id} : \Delta^n \times I \to \operatorname{Im} \alpha \times I$ which induces

$$(\alpha \times \mathrm{Id})_{n+1} : S_{n+1}(\Delta^n \times I) \to S_{n+1}(\operatorname{Im} \alpha \times I)$$

and it is not hard to verify the formula

$$(\alpha \times \mathrm{Id})_{n+1}(p_n) = p_n(\alpha).$$

Furthermore, since $p_n(\alpha)$ has been defined for any affine n-simplex, we may define it by linearity on any linear combination of affine n-simplices. Such a linear combination may appear as an element of $S_n(X)$ for an appropriate subspace X of some Euclidean space. In particular,

$$p_n(\partial_{n+1} \Delta^{n+1}) = \sum_{i=0}^{n+1} (-1)^i p_n(\epsilon_i^{n+1}) \in S_{n+1}(\Delta^{n+1} \times I).$$

Include negative n under this rubric by setting such $p_n = 0$.

Proposition 6.7 (Standard Prism Lemma). *In $S_n(\Delta^n \times I)$, we have*

$$\partial_{n+1} p_n = \overline{\Delta}^n - \Delta^n - p_{n-1}(\partial_n \Delta^n).$$

This basically asserts that the boundary of the standard prism is what we expect it to be, but it includes the signs necessary to get the orientations right.

PROOF. For $n \geq 1$, we have

$$\begin{aligned}\partial_{n+1} p_n &= \sum_{i=0}^{n} (-1)^i \partial_{n+1} [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n] \\ &= \sum_{i=0}^{n} (-1)^i \Big(\sum_{j=0}^{i} (-1)^j [\mathbf{e}_0, \ldots, \hat{\mathbf{e}}_j, \ldots \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n] \\ &\qquad + \sum_{j=i}^{n} (-1)^{j+1} [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \hat{\overline{\mathbf{e}}}_j, \ldots, \overline{\mathbf{e}}_n] \Big).\end{aligned}$$

First separate out the terms where $i = j$. These form the collapsing sum

$$\begin{aligned}\sum_{i=0}^{n} (-1)^{2i} ([\mathbf{e}_0, \ldots, \mathbf{e}_{i-1}, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n] - [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_{i+1}, \ldots, \overline{\mathbf{e}}_n]) \\ = [\overline{\mathbf{e}}_0, \ldots, \overline{\mathbf{e}}_n] - [\mathbf{e}_0, \ldots, \mathbf{e}_n] = \overline{\Delta}^n - \Delta^n.\end{aligned}$$

In the remaining sum, fix a j and consider all terms with that j and $i \neq j$. Write these in the order

$$\begin{gathered}(-1)^{j+1} \sum_{i=0}^{j-1} (-1)^i [\mathbf{e}_0, \ldots, \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \hat{\overline{\mathbf{e}}}_j, \ldots, \overline{\mathbf{e}}_n]) \\ +(-1)^{j+1} \sum_{i=j+1}^{n} (-1)^{i-1} [\mathbf{e}_0, \ldots, \hat{\mathbf{e}}_j, \ldots \mathbf{e}_i, \overline{\mathbf{e}}_i, \ldots, \overline{\mathbf{e}}_n].\end{gathered}$$

(We have also reorganized the placement of various signs.) This adds up to

$$(-1)^{j+1} p_{n-1}(\epsilon_j^n) = -(-1)^j p_{n-1}(\epsilon_j^n).$$

Now add up for all $j = 0, \ldots, n$ to get

$$-\sum_{j=0}^{n} (-1)^j p_{n-1}(\epsilon_j^n) = -p_{n-1}(\sum_{j=0}^{n} (-1)^j \epsilon_j^n) = -p_{n-1}(\partial_n \Delta^n)$$

as required.

We leave it for the student to check this explicitly for $n = 0$. $\square$

We may now carry this over to the spaces X and Y as follows. For a singular n-simplex $\sigma : \Delta^n \to X$, define

$$T_n(\sigma) = (F \circ (\sigma \times \mathrm{Id}))_{n+1}(p_n) \in S_{n+1}(Y).$$

Extending by linearity yields a homomorphism $T_n : S_n(X) \to S_{n+1}(Y)$. Also extend this to $n < 0$ by letting it be zero.

PROPOSITION 6.8 (Chain Homotopy Lemma). *We have the following equality of homomorphisms* $S_n(X) \to S_{n+1}(Y)$

$$\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n = g_n - f_n.$$

This may also be written as a single equation encompassing all n

$$\partial \circ T_\sharp + T_\sharp \circ \partial = g_\sharp - f_\sharp.$$

The proof is given below, but first notice that we now have a proof of the following important result.

THEOREM 6.9 (Homotopy Axiom). *Let* $f, g : X \to Y$ *be homotopic maps. Then for each* n, $H_n(f) = H_n(g) : H_n(X) \to H_n(Y)$.

This may also be written $f_* = g_* : H_*(X) \to H_*(Y)$.

PROOF. Let c_n be a singular n-cycle representing some element of $H_n(X)$. By the above Lemma, since $\partial_n c_n = 0$, we have

$$\partial_{n+1}(T_n(c_n)) = g_n(c_n) - f_n(c_n).$$

It follow that the two terms on the right represent the same element of $H_n(Y)$. From the definition of the induced homomorphisms $H_n(g)$ and $H_n(f)$, it follows that they are equal. $\square$

PROOF OF LEMMA 6.8. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. We have

$$\begin{aligned}
\partial_{n+1}(T_n(\sigma)) &= \partial_{n+1}(F_{n+1}((\sigma \times \mathrm{Id})_{n+1}(p_n))) \\
&= F_n((\sigma \times \mathrm{Id})_n(\partial_{n+1}(p_n))) \\
&= F_n((\sigma \times \mathrm{Id})_n(\overline{\Delta}^n - \Delta^n - p_{n-1}(\partial_n \Delta^n))).
\end{aligned}$$

Expand this out and consider each of the three terms in succession. In doing this, recall that each argument on which the functions are evaluated is in fact a chain in some chain group and as such is a linear combination of maps. For example, the first term is really the map $F \circ (\sigma \times \mathrm{Id}) \circ \overline{\Delta}^n$ from the *space* Δ^n to Y. However, $\overline{\Delta}^n : \Delta^n \to \Delta^n \times I$ is given by $\mathbf{x} \mapsto (\mathbf{x}, 1)$, so the map describing the first term is $g \circ \sigma$ or

$g_n(\sigma)$. Similarly, the second term is $f_n(\sigma)$. For the third term, we have

$$\begin{aligned}
F_n((\sigma \times \mathrm{Id})_n(p_{n-1}(\partial_n \Delta^n))) &= \sum_{i=0}^{n} (-1)^i F_n((\sigma \times \mathrm{Id})_n(p_{n-1}(\epsilon_i^n))) \\
&= \sum_{i=0}^{n} (-1)^i F_n((\sigma \times \mathrm{Id})_n((\epsilon_i^n \times \mathrm{Id})_n(p_{n-1})))) \\
&= \sum_{i=0}^{n} (-1)^i F_n((\sigma \circ \epsilon_i^n \times \mathrm{Id})_n((p_{n-1}))) \\
&= \sum_{i=0}^{n} (-1)^i T_{n-1}(\sigma \circ \epsilon_i^n) \\
&= T_{n-1}(\sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^n) = T_{n-1}(\partial_n \sigma).
\end{aligned}$$

(Make sure you understand the reasons for each step!) □

COROLLARY 6.10. *If $f : X \to Y$ is a homotopy equivalence, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

(We may abbreviate this by saying simply $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.)

PROOF. Choose $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identities. Then $g_* \circ f_*$ and $f_* \circ g_*$ are the respective identities of homology groups and g_* is the inverse of f_*. □

COROLLARY 6.11. *Let X be a contractible space. Then $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbf{Z}$.*

PROOF. X is homotopy equivalent to a point space, and every point space has the indicated homology. □

3.3. More Homological Algebra. Let $f, g : C \to C'$ be morphisms of chain complexes. A *chain homotopy* from f to g is a collection of homomorphisms $T_n : C_n \to C'_{n+1}$ such that

$$d_{n+1} \circ T_n + T_{n-1} \circ d_n = g_n - f_n$$

for each n. From the above discussion, we see that homotopic maps of spaces induce chain homotopic homomorphisms of singular chain complexes. The following algebraic analogue of the homotopy axiom is easy to prove.

PROPOSITION 6.12. *If $f, g : C \to C'$ are chain homotopic morphisms of chain complexes, then they induce the same homomorphisms $H_n(C) \to H_n(C')$ of homology groups of these chain complexes.*

PROOF. Look at the proof in the case of the singular complex of a space. □

Let X be a contractible space; in particular suppose $\mathrm{Id}_X \sim \epsilon$ where $\epsilon : X \to X$ maps everything to a single point p. Then the theory above shows there is a chain homotopy consisting of homomorphisms $T_n : S_n(X) \to S_{n+1}(X)$ such that $\partial_{n+1}T_n + T_{n-1}\partial_n = \epsilon_n - \mathrm{Id}_n$ for each n. Unfortunately, even though ϵ is constant, the homomorphism ϵ_n induced by ϵ in each dimension won't generally be the trivial homomorphism. However, it is possible to remedy this situation by choosing homomorphisms $D_n : S_n(X) \to S_{n+1}(X)$ such that $\partial_{n+1}D_n + D_{n-1}\partial_n = -\mathrm{Id}_n$ for $n > 0$ and $\partial_1 D_0 = \epsilon_0 - \mathrm{Id}_0$ where ϵ_0 sends every 0-simplex to the 0-simplex at p. Such a collection of D_n is called a *contracting chain homotopy*. The existence of such a contracting chain homotopy implies directly that $H_n(X) = 0$ for $n > 0$ and $H_0(X) = \mathbf{Z}$.

4. The Exact Homology Sequence– the Jill Clayburgh Lemma

4.1. Some Homological Algebra. Let C be a chain complex and C' a subcomplex, i.e., C'_n is a subgroup of C_n for each n and $d_n(C'_n) \subseteq C'_{n-1}$ for each n. (The last condition can be stated simply $d(C') \subseteq C'$.) We form a chain complex C/C' called the *quotient chain complex* as follows. Put $(C/C')_n = C_n/C'_n$ and define $d''_n(\bar{c}_n) = \overline{d_n c_n}$ where, as usual, $\bar{c}_n = c_n + C'_n$ denotes the coset of c_n. It is not hard to check that the definition of d'' does not depend on the choice of coset representative. Generally, we shall say that a sequence of chain complexes

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$$

is exact if

$$0 \to C'_n \to C_n \to C''_n \to 0$$

is an exact sequence of groups for each n. According to this definition, if C' is a subcomplex of C, then

$$0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' = C/C' \to 0,$$

where i is the inclusion monomorphism and j is the epimorphism on to the quotient complex, is an exact sequence of chain complexes.

PROPOSITION 6.13. *Given an exact sequence of chain complexes, we have the induced homomorphisms*

$$H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(j)} H_n(C'')$$

for all n. The above sequence is exact, i.e., $\operatorname{Ker} H_n(j) = \operatorname{Im} H_n(i)$.

PROOF. Since $j_n \circ i_n = 0$, it follows that $H_n(j) \circ H_n(i) = 0$ and the left hand side of the proposed equality contains the right hand side. For the reverse inclusion, let c be a cycle which represents an element of $\gamma \in H_n(C) = Z_n(C)/B_n(C)$. Supposing $H_n(j)(\gamma) = 0$, it follows that $j_n(c) \in B_n(C'')$, i.e.,

$$j_n(c) = d_{n+1}c'' = d_{n+1}j_{n+1}(x) = j_n(d_{n+1}x)$$

for some $x \in C_{n+1}$. Hence, $j_n(c - d_{n+1}x) = 0$. It follows that $c - d_{n+1}x = i_n c'$ for some $c' \in C'_n$. However, c' is a cycle, since i_n is a monomorphism and $i_n(d_n c') = d_n(i_n c') = d_n(c - d_{n+1}x) = 0$. This says that the homology class of c' maps to the homology class of $c - dx$ which is the same as the homology class of c. □

More importantly, we imbed the above sequence in a *long exact sequence*

$$\cdots \to H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(j)} H_n(C'')$$
$$\xrightarrow{\partial_n} H_{n-1}(C') \xrightarrow{H_{n-1}(i)} H_{n-1}(C) \xrightarrow{H_{n-1}(j)} H_{n-1}(C'') \to \dots$$

where $\partial_n : H_n(C'') \to H_n(C')$ is a collection of homomorphisms called *connecting homomorphims* which we now define. (The perceptive reader will notice a potential conflict of notation when we apply this to the singular chain complex of a space where the boundary homomorphisms have also been denoted by '∂'. Such conflicts are inevitable if one wants a notation which will act as a spur to the memory in involved situations. Usually they don't cause any problem if one keeps the context straight.) Let $c'' \in Z_n(C'')$ represent an element $\gamma'' \in H_n(C'')$. Since j is an epimorphism, we have $c'' = j(c)$ for some $c \in C_n$. Moreover, $j_{n-1}(d_n c) = d_n(j_n(c)) = d_n c'' = 0$. Hence, $d_n c = i_{n-1}c'$ for some $c' \in C'_{n-1}$. Since i_{n-2} is a monomorphism and $i_{n-2}(d_{n-1}c') = d_{n-1}i_{n-1}c' = d_{n=1}(d_n c) = 0$, it follows that c' is a cycle and it represents some element $\gamma' \in H_{n-1}(C')$. Let $\partial_n(\gamma'') = \gamma'$. We leave it to the student to verify that if c'' is changed to a homologous cycle $c'' + d_{n+1}x$, then the above process gives a cycle in C'_{n-1} which is homologous to c'. Hence, the definition of $\partial_n \gamma''$ does not depend on the choice of the cycle c. We also leave it to the student to prove that ∂_n is a homomorphism for each n.

PROPOSITION 6.14. *The above sequence is exact at every place.*

PROOF. In doing the calculations, we shall drop subscripts when dealing with elements and homomorphisms at the chain level.

$\operatorname{Ker} H_{n-1}(i) = \operatorname{Im} \partial_n$:

Let $c'' \in Z_n(C'')$ represent an element $\gamma'' \in H_n(C'')$. Then $\partial_n\gamma''$ is represented by $c' \in Z_{n-1}(C')$ where $i(c') = dc, j(c) = c''$. Then $i_{n-1}(\partial_n\gamma'')$ is represented by $i(c')$ which is a boundary. Hence, $H_{n-1}(i) \circ \partial_n = 0$.

We leave it as an exercise for the student to show that every element satisfying $i_{n-1}\gamma' = 0$ is of the form $d_n\gamma''$.

$\operatorname{Ker}\partial_n = \operatorname{Im} H_n(j)$:

We leave it as an exercise for the student to show that $\partial_n \circ j_n = 0$.

Suppose that $d_n\gamma'' = 0$ and $c'' \in Z_n(C'')$ represents γ''. That means if we choose $c \in C_n$ such that $j(c) = c''$, then $dc = i(dx')$ for some $x \in C'_{n-1}$. Then $d(c - ix') = 0$, so $c - ix' \in Z_n(C)$. Also, $j(c - ix') = j(c) = c''$. Let $c - ix'$ represent $\gamma \in H_n(C)$. We have $j_n(\gamma) = \gamma''$. □

A little more formalism provides another way to think of the long exact sequence. We define a *graded abelian group* A to be a collection of abelian groups $A_n, n \in \mathbf{Z}$. (Thus a graded abelian group may be viewed as a complex in which the 'd' homomorphisms are trivial.) A morphism $f : A \to B$ from one graded group to another *of degree* k consists of a collection of homomorphisms $f_n : A_n \to B_{n+k}$. Thus, a chain complex C is a graded group together with a morphism d of degree -1. Similarly, a chain homotopy is a morphism of graded groups of degree $+1$.

Using this language, we may think of the symbols $H_*(C'), H_*(C)$, and $H_*(C'')$ as denoting graded groups with morphisms i_*, j_*, ∂_* of degrees $0, 0, -1$ respectively. Then, the long exact sequence may be summarized by the exact triangular diagram

$$\begin{tikzcd}
H_*(C') \arrow[rr, "H_*(i)"] & & H_*(C) \arrow[dl, "H_*(j)"] \\
& H_*(C'') \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

Note that the linear long exact sequence relating the component groups may be thought of as covering this diagram just as the real line covers the circle as its universal covering space.

We now apply the above algebra to singular homology. Let X be a space and A a subspace. Let $i : A \to X$ denote the inclusion. Clearly, we may identify $S_*(A)$ with the subcomplex $i_\sharp(S_*(A))$ of $S_*(X)$. Define the quotient chain complex

$$S_*(X, A) = S_*(X)/S_*(A),$$

i.e., $S_n(X, A) = S_n(X)/S_n(A)$ and $\partial_n : S_n(X, A) \to S_{n-1}(X, A)$ is the quotient of $d_n : S_n(X) \to S_{n-1}(X)$. We call $S_*(X, A)$ the *relative singular chain complex*. Its homology groups are called the *relative singular homology groups* and denoted

$$H_n(X, A).$$

The geometric significance of the relative homology groups is a bit murky. It has something to do with the homology of the quotient space X/A, but they are not exactly the same. $S_n(X, A)$ has as a basis the set of cosets of all singular n-simplices which do not have images in A. (Why?) However, two singular n-simplices $\sigma \neq \sigma'$ in X could project to the same singular n-simplex in X/A. Thus, there is no simple relation between $S_n(X, A)$ and $S_n(X/A)$. We shall see later that in certain cases, $H_n(X, A)$ is indeed the same as $H_n(X/A)$.

The short exact sequence of chain complexes

$$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$$

induces a long exact homology sequence

$$\begin{aligned} \dots \to H_n(A) \xrightarrow{i_n} H_n(X) \to H_n(X, A) & \\ \xrightarrow{\partial_n} H_{n-1}(A) \to H_{n-1}(X) &\to H_{n-1}(X, A) \to \dots \end{aligned}$$

The homomorphism $\partial_n : H_n(X, A) \to H_{n-1}(A)$ has a simple naive geometric interpretation. A cycle in $S_n(X, A)$ may be viewed as a n-chain in X with boundary in A. Such a boundary defines an $(n-1)$-cycle in A.

The relative singular homology groups may be thought of as generalizations of the (absolute) singular homology groups. Indeed, if we take $A = \emptyset$ to be the empty subspace, then $S_n(A)$ is the trivial subgroup of $S_n(X)$ for every n, and $H_n(X, A) = H_n(X)$. In addition, all the properties we have discussed for absolute groups also hold for relative groups, but there are some minor changes that need to be made. For example, if X is a point space, then $H_n(X, A) = 0$ for $n > 0$ for either of the two possible subspaces $A = X$ or $A = \emptyset$, but $H_0(X, X) = 0$. Similarly, if X is a disjoint union of path connected subspaces X_a and A is a subspace of X, then it is not too hard to see that $H_n(X, A) \cong \oplus_a H_n(X_a, X_a \cap A)$, but if X is path connected, we have $H_0(X, A) = 0$ for $A \neq \emptyset$. (Use the long exact sequence and the fact that $H_0(A) \to H_0(X)$ is an isomorphism in that case.) The functoriality holds without qualification. Namely, if $f : (X, A) \to (Y, B)$ is a map of pairs, then $f_\sharp(S_*(A)) \subseteq S_*(B)$, so $f_\sharp$ induces a chain morphism $S_*(X, A) \to S_*(Y, B)$ which in turn induces a homomomorphism

$f_* : H_*(X, A) \to H_*(Y, B)$. It is not hard to check that this provides a functors H_n from the category of pairs of topological spaces to the category of abelian groups (or if you prefer a single functor to the category of graded groups).

The student should verify at least some of the assertions made in the above paragraph, although none of them is particularly startling.

The homotopy axiom for relative singular homology requires a bit more discussion. Let $f, g : (X, A) \to (Y, B)$ be maps of pairs of spaces. A *relative homotopy* from f to g is a map $F : X \times I \to Y$ which is a homotopy from f to g in the ordinary sense and which also satisfies the condition $F(A \times I) \subseteq B$. Note that this implies that the restriction $F' : A \times I \to B$ of F is a homotopy of the restrictions $f', g' : A \to B$ of f, g. It is not generally true, however, that if $f \sim g$ and $f' \sim g'$ that there is a relative homotopy from f to g. (Try to find a counterexample and insert it here in your notes!)

PROPOSITION 6.15 (Relative Homotopy Axiom). *Let $f, g : (X, A) \to (Y, B)$ be relatively homotopic. Then $f_* = g_* : H_*(X, A) \to H_*(Y, B)$.*

PROOF. We just have to follow through the proof in the absolute case and see that everything relativizes properly. By definition, $T_n(\sigma) = (F \circ (\sigma \times \mathrm{Id}))_{n+1}(p_n)$ for σ a singular n-simplex in X. If σ is in fact a singular n-simplex in A, then by the hypothesis, the image of $F \circ (\sigma \times \mathrm{Id})$ is in B. It follows that $T_n(\sigma)$ lies in $S_{n+1}(B)$. Thus, T_n induces a homomorphism of quotients $\overline{T}_n : S_n(X, A) \to S_{n+1}(Y, B)$, and the formula

$$\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n = g_n - f_n$$

projects onto the corresponding formula for $\overline{T}_n$. It then follows as before that f and g induce the same homomorphism of relative homology. □

The *existence of the long exact homology sequence* (which relates absolute and relative homology) is one of the *axioms* we shall use to derive properties of a homology theory. However, there is one important aspect of this sequence we haven't mentioned. Namely, the connecting homomorphisms $\partial_n : H_n(X, A) \to H_n(A)$ are 'natural' in the following sense.

PROPOSITION 6.16 (Naturality of the Connecting Homomorphism). *Let $f : (X, A) \to (Y, B)$ be a map of pairs. Let $f' : A \to B$ be the* *restriction of f. Then the following diagram commutes*

$$\begin{CD} H_n(X,A) @>{\partial_n}>> H_{n-1}(A) \\ @V{f_n}VV @VV{f'_n}V \\ H_n(Y,B) @>{\partial_n}>> H_{n-1}(B) \end{CD}$$

commutes.

PROOF. This follows from the homological algebra below. □

4.2. More Homological Algebra.

PROPOSITION 6.17. *Let*

$$\begin{CD} 0 @>>> C' @>>> C @>>> C'' @>>> 0 \\ @. @V{f'}VV @V{f}VV @VV{f''}V @. \\ 0 @>>> D' @>>> D @>>> D'' @>>> 0 \end{CD}$$

be a commutative diagram of chain complexes. Then the induced diagrams

$$\begin{CD} H_n(C'') @>{\partial_n}>> H_{n-1}(C') \\ @V{f''_n}VV @VV{f'_{n-1}}V \\ H_n(D'') @>{\partial_n}>> H_{n-1}(D') \end{CD}$$

commute.

PROOF. Let $\gamma'' \in H_n(C'')$, and let $c'' \in Z_n(C'')$ represent it. Choose $c \in C_n$ which maps to c'' and $c' \in Z_{n-1}(C')$ which maps to dc. $f'(c')$ represents $f'_{n-1}(\partial_n(\gamma''))$. Because the diagram of chain complexes is commutative, $f(c) \in D_n$ maps to $f''(c'')$ and $f'(c') \in Z_{n-1}(D')$ maps to $df(c) = f(dc)$. This says that $f'(c')$ also represents $\partial_n(f''_n(\gamma''))$. □

The *long exact sequence axiom* should now be taken to assert the existence of connecting homomophisms with the above naturailty property and such that the long homology sequence is exact.

The structure of singular homology theory can be made a bit cleaner by treating it entirely as a functor on the category of pairs (X, A) of spaces and maps of such. As noted above, the absolute singular homology groups are included in this theory by considering pairs of the form $(X, \emptyset)$. To make this theory look a bit more symmetric, one may extend the the long exact sequence to pairs as follows. Suppose we have spaces $A \subseteq B \subseteq X$. Then, it is not too hard to derive an exact homology sequence

$$\begin{tikzcd}
H_*(B, A) \arrow[rr] & & H_*(X, A) \arrow[dl] \\
& H_*(X, B) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

where $\partial_* : H_*(X, B) \to H_*(B, A)$ is a morphism of degree -1 and natural in an obvious sense. We leave this derivation as an exercise for the student.

As we saw above in our discussion of the dimension axiom, the dimension $n = 0$ tends to create exceptions and technical difficulties. Another example of this is the comparison between $H_n(X)$ and $H_n(X, \{P\})$ where P is a point of X. Since $H_n(\{P\}) = 0$ for $n > 0$, the exact homology seqeunce

$$\to H_n(\{P\}) = 0 \to H_n(X) \to H_n(X, \{P\}) \to H_{n-1}(\{P\}) = 0 \to$$

shows that $H_n(X) \cong H_n(X, \{P\})$ for $n > 1$. It is in fact true that this is true also for $n = 1$. This follows from the fact that $H_0(\{P\}) \to H_0(X)$ is a monomorphism, which we leave as an exercise for the student. However, another approach to this issue to to introduce the *reduced homology groups*. These are defined as follows. Consider the unique map $X \to \{P\}$ to any space consisting of a single point. Define $\tilde{H}_*(X) = \operatorname{Ker} H_*(X) \to H_*(\{P\})$. It is not too hard to see that this group does not depend on which particular single point space you use. Also, it is consistent with induced homomorphisms, so it provides a functor on the category of spaces. Note that $\tilde{H}_n(X) = H_n(X)$ for $n \neq 0$, so only in dimension zero is there is a difference. The reduced homology groups give us a way to avoid exceptions in dimension zero. Indeed, $\tilde{H}_0(\{P\}) = 0$ so the dimension axiom may be rephrased by asserting that all the reduced homology groups of a point space are trivial. Similarly, all the reduced homology groups of a contractible space are trivial. Finally, it is possible to show that the exact homology sequence for a pair remains valid if we replace $H_*(X)$ and $H_*(A)$ by the corresponding reduced groups. Of course, this only makes a difference at the tail end of the sequence

$$\to H_1(X, A) \xrightarrow{\partial_1} \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X, A) \to 0.$$

(This includes the assertion that $\operatorname{Im} \partial_1 \subseteq \tilde{H}_0(A) = \operatorname{Ker} H_0(A) \to H_0(\{P\})$.) Using this exact sequence and the fact that $\tilde{H}_0(\{P\}) = 0$, we see that we have quite generally $\tilde{H}_n(X) \cong H_n(X, \{P\})$ for any non-empty space X.

5. Excision and Applications

We now come to an important property called the *excision axiom*. This is quite powerful and it will allow us finally to calculate some interesting homology groups, but it is somewhat technical. Its significance will become clear as we use it in a variety of circumstances. The excision axiom says that given a space X and subspace A, we can 'cut out' subsets of A which are not too large without changing relative homology. We would like to be able to 'cut out', all of A, but in general that is not possible.

THEOREM 6.18 (Excision Axiom). *Let $A \subseteq X$ be spaces. Suppose U is a subspace of A with the property that $\overline{U}$, the closure of U, is contained in A°, the interior of A. Then the inclusion $(X-U, A-U) \to (X, A)$ induces an isomorphism*

$$H_n(X-U, A-U) \cong H_n(X, A)$$

of relative homology groups for every n.

The proof of this theorem is quite difficult, so we shall defer it until we have derived some important consequences.

Using excision, we can finally calculate some homology groups.

LEMMA 6.19. *Let $A^n = \{\mathbf{x} \in D^n \,|\, 1/2 \le |\mathbf{x}| \le 1\}$. Then for each i and each $n > 0$,*

$$H_i(D^n, A^n) \cong \tilde{H}_{i-1}(S^{n-1}).$$

PROOF. The exact sequence

$$\tilde{H}_i(D^n) = 0 \to H_i(D^n, A^n) \to \tilde{H}_{i-1}(A^n) \to \tilde{H}_{i-1}(D^n) = 0$$

shows that $H_i(D^n, A^n) \cong \tilde{H}_{i-1}(A^n)$. On the other hand, $A^n = S^{n-1} \times I$, so S^{n-1} is a deformation retract of A^n and $\tilde{H}_{i-1}(A^n) \cong \tilde{H}_{i-1}(S^{n-1})$. □

THEOREM 6.20. *We have*

$$H_0(S^n) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{if } n = 0 \\ \mathbf{Z} & \text{if } n > 0. \end{cases}$$

For $i > 0$,

$$H_i(S^n) = \begin{cases} 0 & \text{if } 0 < i < n \text{ or } i > n \\ \mathbf{Z} & \text{if } i = n. \end{cases}$$

PROOF. The zero sphere S^0 consists of two points and the conclusion is clear for it. Note that $\tilde{H}_0(S^0) \cong \mathbf{Z}$ and in fact it is easy to see that it consists of all elements of the form $(n, -n)$ in $\mathbf{Z} \oplus \mathbf{Z}$. (Just consider the kernel of $H_0(S^0) \to H_0(\{P\}) = \mathbf{Z}$.)

The remaining statements amount to the assertion that for $n > 0$, $\tilde{H}_i(S^n) = 0$ except for $i = n$ in which case we get $\mathbf{Z}$. To prove this latter assertion, proceed as follows. Let $U^n = \{\mathbf{x} \in S^n \mid -1 \leq x_{n+1} < 0\}$ and let $B_+^n = \{\mathbf{x} \in S^n \mid -1 \leq x_{n+1} \leq 1/2\}$. By excision, we have $H_i(S^n - U^n, B^n - U^n) \cong H_i(S^n, B^n)$. However, the pair $(D_+^n = S^n - U^n, A_+^n = B^n - U^n)$ is clearly homeomorphic to the pair (D^n, A^n). It follows from this and the above lemma that $H_i(S^n, B_n) \cong H_{i-1}(S^{n-1})$. On the other hand, since B^n is contractible, the long exact sequence for the pair (S^n, B^n) shows that $\tilde{H}_i(S^n) \cong H_i(S^n, B^n)$ for all i. Putting these facts together, we get

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}) \cong \ldots \cong \tilde{H}_{i-n}(S^0).$$

Since the last term is zero if $i \neq n$ and $\mathbf{Z}$ if $i = n$, we are done. □

Important Note on the Proof. There is one fact which can be squeezed out of this proof which we shall need later. Namely, consider the map $r^n : S^n \to S^n$ which sends $x_0 \to -x_0$ and fixes all other coordinates. (This is the reflection in the hyperplane perpendicular to the x_0 axis.) If you carefully check all the isomorphisms in the above proof, you will see that they may be chosen to be consistent with the maps r^n, i.e., so that the diagram

$$\begin{tikzcd}
\tilde{H}_i(S^n) \arrow[r, "\cong"] \arrow[d, "r_*^n"'] & H_{i-1}(S^{n-1}) \arrow[d, "r_*^{n-1}"] \\
H_i(S^n) \arrow[r, "\cong"'] & H_{i-1}(S^{n-1})
\end{tikzcd}$$

commutes. You should go through all the steps and check this for yourself. It is a good exercise in understanding where all the isomorphisms used in the proof come from.

Once we have calculated the homology groups of spheres, we may derive a whole lot of consequences.

THEOREM 6.21. *(a) S^n and S^m for $n \neq m$ do not have the same homotopy type.*

(b) R^n and R^m for $n \neq m$ are not homeomorphic.

(c) S^n for $n > 1$ is a space which is simply connected but not contractible.

PROOF. You should be able to figure these out. □

The Brouwer Fixed Point Theorem now follows exactly as proposed in the first introduction.

THEOREM 6.22 (Brouwer). *Any map $f : D^n \to D^n, n > 0$, has a fixed point.*

PROOF. Go back and look at the introduction. The nonexistence of a fixed point allowed us to construct a retraction of D^n onto S^{n-1} whence $H_n(S^n) = \mathbf{Z}$ is a direct summand of $H_n(D^n) = 0$. □

5.1. Degree and Vector Fields on Spheres. Let $f : S^n \to S^n$. We call such a map a *self–map.* It induces a homomorphism $f_* : H_n(S^n) = \mathbf{Z} \to H_n(S^n) = \mathbf{Z}$. (For completeness, use $\tilde{H}_0$ for $n = 0$.) Such a homomorphism is necessarily multiplication by some integer d which is called the *degree* of the map f and denoted $d(f)$. As in our discussion of the fundamental group, the degree measures in some sense how many times the image of the map covers the sphere.

PROPOSITION 6.23. *Degree has the following properties.*

(a) Given self–maps, f, g of S^n, we have $d(g \circ f) = d(g)d(f)$. Also, the degree of the identity is 1 and the degree of a constant self–map is zero.

(b) Homotopic self maps have the same degree.

(c) Reflection in a hyperplane through the origin has degree -1.

(d) The antipode map has degree $(-1)^{n+1}$.

The converse of (b) is also true, i.e., self–maps with the same degree are homotopic. (This follows from an important theorem called Hopf's Theorem, which you will see later.)

PROOF. Everything in (a) follows by functorality, i.e., $\mathrm{Id}_* = \mathrm{Id}$ and $(g \circ f)_* = g_* \circ f_*$. To see that the degree of a constant map is zero, factor it through a point space.

(b) follows from the homotopy axiom.

(d) follows from (a) and (c). Namely, the antipode map $\mathbf{x} \mapsto -\mathbf{x}$ may be obtained by composing the $n + 1$ component reflections

$$r_k : x_i \mapsto \begin{cases} x_i & i \neq k \\ & \qquad \mapsto -x_i \qquad i = k. \end{cases}$$

Finally, to prove (c), choose a coordinate system so that the hyperplane is given by $x_0 = 0$ and the associated reflection is $x_0 \mapsto -x_0$, with the other coordinates fixed. We noted above that the calculation $H_n(S^n) \cong \tilde{H}_0(S^0) \cong \mathbf{Z}$ is consistent with the reflection (by induction).

However, $\tilde{H}_0(S^0)$ can be identified with the subgroup of $\mathbf{Z} \oplus \mathbf{Z}$ consisting of all $(n, -n)$ where a basis for $\mathbf{Z} \oplus \mathbf{Z}$ may be taken to be the two points $x_0 = 1$ and $x_0 = -1$ in S^0. Clearly, the reflection swtiches these points and so sends $(n, -n)$ to $(-n, n) = -(n, -n)$. □

PROPOSITION 6.24. *Let f, g be self–maps of S^n. If $f(x) \neq g(x)$ for all $x \in S^n$, then $g \sim a \circ f$ where a is the antipode map.*

PROOF. Since $g(x)$ is never antipodal to $-f(x)$, the line connecting them never passes through the origin. Define $G(x, t) = t(-f(x)) + (1 - t)g(x)$, so $G(x, t) \neq 0$ for $0 \leq t \leq 1$. Define $F(x, t) = G(x, t)/|G(x, t)|$. $F(x, 0) = g(x)$ and $F(x, 1) = -f(x) = a(f(x))$. □

A *tangent vector field* T on a sphere S^n is a function $T : S^n \to \mathbf{R}^{n+1}$ such that for each $x \in S^n$, $T(x) \cdot x = 0$. Thus, we may view the vector $T(x)$ sitting at the end of the vector x on S^n and either it is zero or it is tangent to the sphere there. (This is a special case of a much more general concept which may be defined for any differentiable manifold.)

THEOREM 6.25. *There do not exist non-vanishing vector fields T defined on an even dimensional sphere S^{2n}.*

For the case S^2, this is sometimes interpreted as saying something about the possibility of combing hair growing on a billiard ball.

PROOF. Assume T is a nonvanishing tangent vector field on S^{2n}. Let $t(x) = T(x)/|T(x)|$. Since $t(x) \perp x$, we certainly never have $t(x) = x$. It follows from the propostion that $t \sim a \circ \mathrm{Id} = a$. Hence, $d(t) = (-1)^{2n+1} = -1$. However, it is also true that $t(x) \neq -x$ for the same reason, so $t \sim a \circ a = \mathrm{Id}$. Thus, $d(t) = 1$. Thus, we have a contradiction to the assumption that T never vanishes. □

Odd dimensional spheres do have non-vanishing vector fields. For example, for $n = 1$, we have $T(x_0, x_1) = (-x_1, x_0)$.

Similarly for odd $n > 1$, define

$$T(x_0, \dots, x_n) = (-x_1, x_0, -x_3, x_2, \dots, -x_n, x_{n-1}).$$

One can ask how many *linearly independent* vector fields there are on S^{2n+1} as a function of n. A lower bound was established by Hurwitz and Radon in the 1920s, but the proof that this number is also an upper bound was done by J. F. Adams in 1962 using K-theory.

6. Proof of the Excision Axiom

Suppose $\overline{U} \subseteq A^\circ$ as required by the excision axiom. Consider the inclusion monomorphism $i_\sharp : S_*(X - U) \to S_*(X)$. Since a singular simplex has image both in $X - U$ and in A if and only if its image is in $A - U$, it follows that $S_*(X - U) \cap S_*(A) = S_*(A - U)$. By basic group theory, this tells us that $i_\sharp$ induces a monomomorphism of quotients

$$i_\sharp : S_*(X - U, A - U) \to S_*(X, A).$$

The image of this monomorphism is

$$S'_*(X, A) = (S_*(X - U) + S_*(A))/S_*(A)$$

which is a subcomplex of $S_*(X)/S_*(A) = S_*(X, A)$. We shall show below that the inclusion $S'_*(X, A) \to S_*(X, A)$ induces an isomorphism in homology. Putting this together with $i_\sharp$, we see that the induced homomorphism $H_*(X - U, A - U) \to H_*(X, A)$ is an isomorphism.

We shall accomplish the desired task by proving something considerably more general. Let $\mathcal{U}$ denote a collection of subspaces $\{U\}$ of X such that the interiors $\{U^\circ\}$ *cover* X. In the above application, the collection consists of two sets A and $X - U$. (Under the excision hypothesis, we have

$$X = (X - \overline{U}) \cup A^\circ$$

so the interiors cover.) Consider the subcomplex $S^{\mathcal{U}}_*(X)$ with basis all singular simplices with images in some subspace U in $\mathcal{U}$. Such singular chains are called *$\mathcal{U}$-small*. Similarly, let $S^{\mathcal{U}}_*(A)$ be the corresponding subcomplex for A and $S^{\mathcal{U}}_*(X, A)$ the resulting quotient complex. Since $S^{\mathcal{U}}_*(A) = S^{\mathcal{U}}_*(X) \cap S_*(A)$, it follows that $S^{\mathcal{U}}_*(X, A)$ may be identified with a subcomplex of $S_*(X, A)$. We shall show in all these cases that the inclusion $S^{\mathcal{U}}_*(-) \to S_*(-)$ induces an isomorphism in homology. The student should verify that in the above application

$$\begin{aligned} S^{\mathcal{U}}_*(X) &= S_*(X - U) + S_*(A) \\ S^{\mathcal{U}}_*(A) &= S_*(A) \\ S^{\mathcal{U}}_*(X, A) &= S'_*(X, A). \end{aligned}$$

In order to prove the one isomorphism we want, that in the relative case, it suffices to prove the isomomorphisms for X and A. For, the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & S_*^{\mathcal{U}}(A) \arrow[r] \arrow[d] & S_*^{\mathcal{U}}(X) \arrow[r] \arrow[d] & S_*^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & S_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

induces a diagram in homology

$$\begin{tikzcd}
\cdots \arrow[r] & H_n^{\mathcal{U}}(A) \arrow[r] \arrow[d] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d] & H_n^{\mathcal{U}}(X, A) \arrow[r] \arrow[d] & H_{n-1}^{\mathcal{U}}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Suppose we have shown that the vertical arrows between the absolute groups are isomorphisms. Then it follows from the following result that the vertical arrows between the relative groups are isomorphisms as well.

PROPOSITION 6.26 (Five Lemma). *Suppose we have a commutative diagram of abelian groups*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

with exact rows. If f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism.

PROOF. You will have to do this yourself. It is just a tedious diagram chase. See the Exercises where you will be asked to prove something slightly more general, the so-called 'Four Lemma'. $\square$

It now follows that we need only prove the desired isomorphism in the absolute case and then apply it separately to X and to A. The rest of this section will be concerned with that task.

6.1. Barycentric Subdivision. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Our approach will be to subdivide Δ^n into affine sub-simplices of small enough diameter such that σ carries each into at least one set in the covering. We know this is possible if the sets are small enough by the Lebesgue Covering Lemma since $\sigma(\Delta^n)$ is compact. Since the

Lebesgue Covering Lemma requires an open covering, we need to assume the interiors of the sets cover X.

The subdivision accomplished by iterating the process of *barycentric subdivision* which we now describe. We define this in general for affine n-simplices by induction. First, we need some notation Given an affine n-simplex $\alpha = [\mathbf{x}_0, \dots, \mathbf{x}_n]$, let

$$\mathbf{b}(\alpha) = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{x}_i$$

denote the *barycenter* of the vertices. Generally, let $a = \sum_j \alpha_j$ be an affine p-chain, i.e., a linear combination of affine p-simplices. If $\mathbf{b}$ is any point not in the affine subspaces spanned by the vertices of any α_j in a, then let $[\mathbf{b}, a] = \sum_j [\mathbf{b}, \alpha_j]$.

We define an operator Sd which associates with each affine n-chain a another affine n-chain $Sd_n(a)$. We do this by defining it on affine simplices and extending by linearity. For a zero-simplex $\alpha = [x_0]$, let $Sd_0(\alpha) = \alpha$. Assume Sd_{n-1} has been defined, and let α be an affine n-simplex. Define

$$Sd_n(\alpha) = [\mathbf{b}(\alpha), Sd_{n-1}(\partial_n \alpha)].$$

The n-simplices occuring in $Sd_n\alpha$ constitute the barycentric subdivision of α. We claim that this subdivision operator is consistent with the boundary operator for affine simplices. Namely, if $\alpha = [\mathbf{x}_0, \dots, \mathbf{x}_n]$, then abbreviating $\mathbf{b} = \mathbf{b}(\alpha)$ and $\alpha_i = \alpha \circ \epsilon_i^n$ (the ith face of α), we have

$$\partial_n(Sd_n(\alpha)) = \partial_n[\mathbf{b}, Sd_{n-1}(\partial_n \alpha)].$$

However, it is not hard to check that for any affine p-chain a we have

$$\begin{aligned} \partial_{p+1}[\mathbf{b}, a] &= a - [\mathbf{b}, \partial_p a] \qquad p > 0 \\ \partial_1[\mathbf{b}, a] &= a - (\tilde{\partial}_0(a)\mathbf{b} \qquad p = 0, \end{aligned}$$

where $\tilde{\partial}_0$ denotes the sum of the coefficients operator. (Just prove the formula for simplices and extend by linearity.) Hence, it follows that for $n > 1$,

$$\partial_n(Sd_n(\alpha)) = \partial_n[\mathbf{b}, Sd_{n-1}(\partial_n \alpha)] = Sd_{n-1}(\partial_n \alpha) - [\mathbf{b}, \partial_{n-1}(Sd_{n-1}(\partial_n \alpha))].$$

By induction, $\partial_{n-1} Sd_{n-1}\partial_n = Sd_{n-2}\partial_{n-1}\partial_n = 0$, so we conclude $\partial_n Sd_n = Sd_{n-1}\partial_n$ as required. The proof for $n = 1$ is the same except we use that $\tilde{\partial}_0\partial_1 = 0$.

We now extend the operator Sd to singular chains in any space by defining for a singular n-simplex $\sigma : \Delta^n \to X$,

$$Sd_n^X(\sigma) = \sigma_\sharp(Sd_n\Delta^n).$$

As usual, define Sd_n to be zero in negative dimensions because that is the only possible definition.

Note first that there is a slight subtlety in the definition of Sd. For an affine n-simplex viewed as a map from the standard simplex, there are two possible definitions of $Sd_n(\alpha)$, either the original definition or $\alpha_\sharp(\Delta^n)$. If you examine the arguments we shall use carefully, we must know that these definitions are the same. We leave that for the student to verify.

Note next that Sd^X has an important naturality property, i.e., if $f : X \to Y$ is a map, then

$$\begin{aligned} f_\sharp(Sd_n^X(\sigma) &= f_\sharp(\sigma_\sharp(Sd_n\Delta^n) \\ &= (f\circ\sigma)_\sharp(Sd_n\Delta^n) \\ &= Sd_n^Y(f\circ\sigma) = Sd_n^Y(f_\sharp(\sigma)) \end{aligned}$$

so

$$f_\sharp \circ Sd^X = Sd^Y \circ f_\sharp.$$

Thirdly, we note that Sd^X commutes with the boundary operator on $S_*(X)$. We leave that as an exercise for the student.

LEMMA 6.27. *Let X be a space. There exist natural homomorphisms $T_n^X : S_n(X) \to S_{n+1}(X)$ such that*

$$\partial_{n+1}T_n^X + T_{n-1}^X\partial_n = Sd_n^X - \mathrm{Id}_n$$

for each n.

In other words, the subdivision operator is chain homotopic to the identity.

PROOF. Rather than trying to define T_n explicitly, we use a more abstract approach. A vast generalization of this method called the *method of acyclic models* allows us to proceed in a similar manner in many different circumstances. We shall go into this method in greater detail later, but for the moment concentrate on how we use the contractibility of certain standard spaces, i.e., affine simplices, to define T_n.

We proceed by induction. Let $T_n^X = 0$ for $n < 0$. For $n = 0$, it is not hard to see that $Sd_0^X = \mathrm{Id}_0$ for any space X, so we may take $T_0^X = 0$. Suppose $n > 0$ and T_p^X has been defined for every $p < n$ and every space X, that it is natural in the sense that if $f : X \to Y$ is a map then $f_\sharp \circ T^X = T^Y \circ f_\sharp$, and that it satifies the desired formula. Let $T = T^{\Delta^n}$, and consider

$$u = Sd_n(\Delta^n) - \Delta^n - T_{n-1}(\partial_n \Delta^n).$$

We have

$$\begin{aligned}\partial_n u &= \partial_n Sd_n(\Delta^n) - \partial_n \Delta^n - \partial_n T_{n-1}(\partial_n \Delta^n)\\ &= Sd_{n-1}(\partial_n \Delta^n) - \partial_n \Delta^n - Sd_{n-1}(\partial_n \Delta^n) + \partial_n \Delta_n + T_{n-2}(\partial_{n-1}\partial_n \Delta^n)\\ &= 0.\end{aligned}$$

Since Δ^n is contractible, its homology in all dimensions $n > 0$ is trivial. Hence, $u = \partial_{n+1} t_n$ for some singular $(n+1)$-chain t_n in $S_{n+1}(\Delta^n)$. By construction,

$$\partial_{n+1} t_n + T_{n-1}(\partial_n \Delta^n) = Sd_n(\Delta^n) - \Delta^n.$$

Now define $T_n^X : S_n(X) \to S_{n+1}(X)$ by

$$T_n^X(\sigma) = \sigma_\sharp(t_n)$$

for singular n-simplices σ in X. As in the case of Sd_n^X, the naturality of T_n^X follows immediately from its definition. (Check this yourself!) The desired formula is verified as follows.

$$\begin{aligned}\partial_{n+1} T_n^X(\sigma) &= \partial_{n+1} \sigma_\sharp t_n = \sigma_\sharp \partial_{n+1} t_n\\ &= \sigma_\sharp Sd_n(\Delta^n) - \sigma_\sharp \Delta^n - \sigma_\sharp T_{n-1}(\partial_n \Delta^n)\\ &= Sd_n^X \sigma_\sharp(\Delta^n) - \sigma_\sharp \Delta^n - T_{n-1}^X \sigma_\sharp(\partial_n \Delta^n)\\ &= Sd_n^X(\sigma) - \sigma - T_{n-1}^X(\partial_n \sigma)\end{aligned}$$

as required. □

Note that it follows easily from this that $(Sd^X)^q$ is chain homotopic to the identity for each q. (See the Exercises.)

We are now ready to prove

THEOREM 6.28. *Let $\mathcal{U}$ be a collection of subspaces with interiors covering X. Then $H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism.*

PROOF. In order to reduce to consideration of $\mathcal{U}$-small chains, we iterate the process of barycentric subdivision. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Denote by $|\sigma|$ its image, also called its *support.*

(Similarly, the support $|c|$ of any chain is the union of the supports of the simplices occuring in it.)

We shall show first that for q sufficiently large, $(Sd^X)^q(\sigma)$ is $\mathcal{U}$-small, i.e., the support $|\tau|$ of every singular simplex τ in $(Sd^X)^q(\sigma)$ is contained in some set of the covering. Since any singular chain involves at most finitely many singular simplices, the same asserstion will then apply to each singular chain. By naturality,

$$(Sd^X)^q(\sigma) = \sigma_\sharp(Sd^q(\Delta^n)).$$

(As usual, for α an affine simplex $Sd(\alpha)$ may be viewed as an affine chain in any convenient subspace of $\mathbf{R}^N$, where N is a convenient integer. In this case, the subspace is Δ^n.) Hence, it suffices to show that for q sufficiently large, the diameter of the support of each affine simplex in $Sd^q(\Delta^n)$ is small enough for the conclusions to be valid by the Lebesgue Covering Lemma. For an affine chain a, this maximum diameter is called the *mesh* of the chain and we shall denote it $m(a)$.

LEMMA 6.29. *If α is an affine n-simplex, then $m(Sd(\alpha)) \leq \dfrac{n}{n+1}m(\alpha)$.*

PROOF. Let $\mathbf{b} = \mathbf{b}(\alpha)$ be the barycenter of $\alpha = [\mathbf{x}_0, \dots, \mathbf{x}_n]$. Then

$$m(Sd(\alpha)) \leq \max_{i=1,\dots,n} |\mathbf{b} - \mathbf{x}_i|.$$

(See the above diagram for an indication of the argument for this, in particular why only the lengths of the edges from $\mathbf{b}$ to the vertices of α need to be considered.) Hence,

$$\begin{aligned} |\mathbf{b} - \mathbf{x}_i| = |\frac{1}{n+1}\sum_j \mathbf{x}_j - \mathbf{x}_i| &= \frac{1}{n+1}|\sum_{j\neq i}(\mathbf{x}_j - x_i)| \\ \leq \frac{1}{n+1}\sum_{j\neq i}|\mathbf{x}_j - \mathbf{x}_i| &\leq \frac{n}{n+1}m(\alpha). \end{aligned}$$

$\square$

It follows that $m(Sd^q(\Delta^n)) \leq (n/n+1)^q m(\Delta^n) \to 0$ as $q \to \infty$. Thus as mentioned above, and immediate consequence of this is that for any singular n-cycle c in X, there is a q such that $(Sd^X)^q c$ is $\mathcal{U}$-small. Since $(Sd^X)^q c$ is homologous to c (using the fact that $(Sd^X)^q$ is chain homotopic to Id), it follows that any singular n-cycle is homologous to one in $S_*^{\mathcal{U}}(X)$.

Note also that by reasoning similar to that above, it is easy to see that if c is a $\mathcal{U}$-small singular chain, then so is $(Sd^X)^q c$ for any q.

Suppose c is $\mathcal{U}$-small, and $c = \partial y$ where $y \in S_*(X)$. (y need note be $\mathcal{U}$-small. Then $(Sd^X)^q c = \partial (Sd^X)^q y)$, and $(Sd^X)^q y$ will be $\mathcal{U}$-small if q is large enough by the above Lemma. Let T be the natural chain homotopy introduced above, i.e., $T_n(\sigma) = \sigma_\sharp(t_n)$ where $t_n \in S_{n+1}(\Delta^n)$. We shall show that $(Sd^X)^q c$ is homologous to c by means of the boundary of a $\mathcal{U}$-small chain. To this end, we write

$$Sd^q - \mathrm{Id} = Sd^{q-1}(Sd - \mathrm{Id}) + Sd^{q-2}(Sd - \mathrm{Id}) + \cdots + Sd - \mathrm{Id},$$

and we see that it suffices to show that $Sd^X c - c$ is the boundary of a $\mathcal{U}$-small chain. Let σ be a $\mathcal{U}$-small simplex in c. We have

$$Sd^X \sigma - \sigma = \partial T^X \sigma + T^X \partial \sigma.$$

When we add up the singular simplices in c, the second terms on the right will add up to zero. Hence, it suffices to show that $T^X \sigma$ is $\mathcal{U}$-small if σ is. However, as above, $T^X(\sigma) = \sigma_\sharp(t_n)$ so the support of every singular simplex appearing in this chain is contained in the support of σ, and we are done. □

7. Relation between π_1 and H_1

THEOREM 6.30. *Let X be a path connected space and x_0 a base point. Then there is a natural isomorphism*

$$h : \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \to H_1(X_0).$$

Here ‘naturality’ means that either for change of basepoint or for maps of spaces with basepoint, the appropriate diagrams are commutative. Think about what that should mean in each case.

The rest of this section is devoted to the proof.

Let σ be any path in X. σ is also a singular 1-simplex. If σ is a loop, it is clear that the corresponding 1-simplex is a cycle. This provides us a map from the loops at x_0 to $Z_1(X)$. Suppose $\sigma \sim_{\dot{I}} \tau$. Then σ is homologous in $S_1(X)$ to τ. For, let $F : I^2 \to X$ be a homotopy from σ to τ which is constant on both vertical edges. The diagram below shows an affine 2-chain a in I^2 with $\partial F_\sharp(a) = F_\sharp(\partial a) = \sigma + \epsilon_{x_2} - \tau - \epsilon_{x_1}$, where x_1, x_2 are the common endpoints of the paths. If σ, τ are loops at x_0, then $x_1 = x_2 = x_0$ and it follows that σ is homologous to τ. It follows that we get a map $\pi_1(X, x_0) \to H_1(X)$. We claim next that this map is a homomorphism. Indeed, the diagram below shows that $\sigma * \tau$ is homologous to $\sigma + \tau$ for any two paths σ, τ for which the

$*$ composition is defined. Since $H_1(X)$ is abelian, we get finally an induced homomorphism $\pi_1/[\pi_1, \pi_1] \to H_1$. Call this homomorphism h.

To show h is an isomorphism, we shall define an inverse homomorphism j. For each point $x \in X$, let ϕ_x be a fixed path from x_0 to x, and assume $\phi_{x_0} = \epsilon_{x_0}$. For σ an singular 1-simplex in X, let

$$\hat{\sigma} = \phi_{x_1} * \sigma * \overline{\phi}_{x_2}$$

where x_1, x_2 are the endpoints of σ. The map $\sigma \mapsto \hat{\sigma}$ defines a homomorphism $S_1(X) \to \pi_1/[\pi_1, \pi_1]$ since the latter group is abelian and the singular 1-simplices form a basis of the former group. Under this homomorphism, boundaries map to the identity. For, let ρ be a singular 2-simplex. Then if $\partial\rho = \sigma_0 - \sigma_1 + \sigma_2$ as indicated below, where σ_0 has vertices x_1, x_2, σ_1 has vertices x_3, x_2, and σ_2 has vertices x_3, x_1.

It follows that $\partial\rho$ maps to the homotopy class of the path

$$\phi_{x_1} * \sigma_0 * \overline{\phi}_{x_2} * \phi_{x_2} * \overline{\sigma}_1 * \overline{\phi}_{x_3} * \phi_{x_3} * \sigma_2 * \overline{\phi}_{x_1} \sim_{\dot{I}} \phi_{x_1} * (\sigma_0 * \overline{\sigma}_1 * \sigma_2) * \overline{\phi}_{x_1}.$$

However, the expression in parentheses is clearly ρ_* of a loop γ in Δ^2 based at e_1. Since Δ^1 is simply connected, it follows that

$$\phi_{x_1} * (\sigma_0 * \overline{\sigma}_1 * \sigma_2) * \overline{\phi}_{x_1} \sim_{\dot{I}} \phi_{x_1} * (\epsilon_{x_1}) * \overline{\phi}_{x_1} \sim_{\dot{I}} \epsilon_{x_0}.$$

If we restrict the homomorphism defined by $\sigma \mapsto \hat{\sigma}$ to the subgroup $Z_1(X)$, since $B_1(X)$ maps to the identity, we get a homomorphism $j : H_1(X) \to \pi_1/[\pi_1, \pi_1]$. We shall show that h and j are inverse homomorphisms.

If σ is a loop at x_0 in X, then $\hat{\sigma} = \sigma$, so it follows that $\sigma \mapsto \sigma \mapsto \hat{\sigma}$ is the identity, i.e., $j \circ h = \mathrm{Id}$.

Suppose $c = \sum_i n_i\sigma_i$ is a singular 1-cycle. Let σ_i have vertices p_i and q_i. Then under $h \circ j$, σ_i maps to $\hat{\sigma}_i$ which by the above is homologous to $\phi_{p_i} + \sigma_i + \overline{\phi}_{q_i}$. (In general, $\alpha * \beta$ is homologous to $\alpha + \beta$.) By the same reasoning, $\overline{\phi}_{q_i}$ is homologous to $-\phi_{q_i}$, so σ_i maps to $\phi_{p_i} + \sigma_i - \phi_{q_i}$. Hence, $c = \sum_i n_i\sigma_i$ maps to

$$\sum_i n_i\sigma_i + \sum_i n_i(\phi_{p_i} - \phi_{q_i}) = c + \sum_i n_i(\phi_{p_i} - \phi_{q_i}).$$

However, $\sum_i n_i(p_i - q_i) = \partial c = 0$, and this implies that $\sum_i n_i(\phi_{p_i} - \phi_{q_i}) = 0$. For, the first sum may be reduced to a linear combination of a set of distinct singular 0-simplices in which the coefficients must necessarily be zero, and the second sum is the same linear combination of a set of distinct singular 1-simplices in one to one correlation with the former set, so it also must be zero.

8. The Mayer-Vietoris Sequence

The Mayer-Vietoris sequence is the analogue for homology of the Seifert–VanKampen Theorem for the fundamental group.

Let $X = X_1 \cup X_2$ where the interiors of X_1 and X_2 also cover X. Let $i_1 : X_1 \cap X_2 \to X_1, i_2 : X_1 \cap X_2 \to X_2$ and $j_1 : X_1 \to X, j_2 : X_2 \to X$ be the inclusion maps. The first pair induces the homomorphism

$$i_* : H_*(X_1 \cap X_2) \to H_*(X_1) \oplus H_*(X_2)$$

defined by $i_*(\gamma) = (i_{1*}(\gamma), -i_{2*}(\gamma))$. The second pair induce the homomorphism

$$j_* : H_*(X_1) \oplus H_*(X_2) \to H_*(X)$$

defined by $j_*(\gamma_1, \gamma_2) = j_{1*}(\gamma_1) + j_{2*}(\gamma_2)$. It is not hard to check that $j_* i_* = 0$. In fact, we shall see that $\operatorname{Ker} j_* = \operatorname{Im} i_*$ and that this is part of a long exact sequence for homology. To this end, we need to invent a connecting homomorphism

$$\partial_* : H_*(X) \to H_*(X_1 \cap X_2)$$

which reduces degree by 1.

There are two ways to do this. First, the covering $X = X_1 \cup X_2$ satisfies the conditions necessary for $H_*^{\mathcal{U}}(X) \to H_*(X)$ to be an isomorphism. In this case, it is not hard to check that

$$S^{\mathcal{U}}(X) = S_*(X_1) + S_*(X_2).$$

Also, there is a homomorphism

$$S_*(X_1) \oplus S_*(X_2) \to S_*(X_1) + S_*(X_2).$$

defined by $(c_1, c_2) \mapsto c_1 + c_2$. The kernel of this homomorphism is easily seen to be the image of

$$S_*(X_1 \cap X_2) \to S_*(X_1) \oplus S_*(X_2)$$

defined by $c \mapsto (i_{1\sharp}(c), -i_{2\sharp}(c))$ so it follows that we have a short exact sequence of chain complexes

$$0 \to S_*(X_1 \cap X_2) \to S_*(X_1) \oplus S_*(X_2) \to S^{\mathcal{U}}(X) \to 0.$$

The long exact homology seqeunce of this sequence is the Mayer–Vietoris sequence.

Another approach is to derive the Mayer-Vietoris sequence from the excision axiom. This has the advantage that we don't need to add

Mayer-Vietoris as an additional 'axiom' in any more abstract approach to homology theories.

Let $U = X_1 - X_1 \cap X_2$ and let $A = X_1$. Then, $\overline{U} \subseteq X_1{}^\circ$. For, let $x \in \overline{U}$. If $x \in X_2{}^\circ$, then there is an open neighborhood W of x contained in X_2, so W cannot intersect $U = X_1 - X_1 \cap X_2$. This contradicts the assertion that $x \in \overline{U}$, so it must be true that $x \notin X_2{}^\circ$. Hence, $x \in X_1{}^\circ \subseteq X_1$ as required.

We have $X - U = X - (X_1 - X_1 \cap X_2) = X_2$ and $X_1 - U = X_1 \cap X_2$. Hence, by excision, the inclusion $h : (X_2, X_1 \cap X_2) \to (X, X_1)$ induces an isomorphsim

$$h_* : H_*(X_2, X_1 \cap X_2) \to H_*(X, X_1).$$

Consider the commutative diagram of long exact sequences

$$\begin{tikzcd}
H_n(X_1 \cap X_2) \arrow[r, "i_{2*}"] \arrow[d, "i_{1*}"'] & H_n(X_2) \arrow[r, "k_*"] \arrow[d, "j_{2*}"'] & H_n(X_2, X_1 \cap X_2) \arrow[r, "\partial"] \arrow[d, "h_*"'] & H_{n-1}(X_1 \cap X_2) \arrow[d, "i_{1*}"'] \\
H_n(X_1) \arrow[r, "j_{1*}"] & H_n(X) \arrow[r, "l_*"] & H_n(X, X_1) \arrow[r] & H_{n-1}(X_1)
\end{tikzcd}$$

Define the homomorphisms i_* and j_* as above and define the desired connecting homomorphism $H_n(X) \to H_{n-1}(X_1 \cap X_2)$ as $\partial_* \circ (h_*)^{-1} \circ l_*$. It is not hard to check that this homomorphism is natural in the obvious sense. We leave it to the student to check this. That the Mayer–Vietoris sequence is exact follows from the following Lemma

Lemma 6.31 (Barratt-Whitehead). *Suppose we have a commutative diagram of abelian groups*

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"'] & B_n \arrow[r, "j_n"] \arrow[d, "g_n"'] & C_n \arrow[r, "k_n"] \arrow[d, "h_n"'] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & {} \\
\cdots \arrow[r] & A'_n \arrow[r, "i'_n"] & B'_n \arrow[r, "j'_n"] & C'_n \arrow[r, "k'_n"] & A'_{n-1} \arrow[r, "i'_{n-1}"] & \cdots \arrow[r] & {}
\end{tikzcd}$$

with exact rows. Suppose h_n is an isomorphism for every n. Then

$$\cdots \longrightarrow A_n \xrightarrow{(f_n, -i_n)} A'_n \oplus B_n \xrightarrow{i'_n + g_n} B'_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \cdots$$

is exact where $\partial = k_n \circ (h_n)^{-1} \circ j'_n$.

Proof. This is just a diagram chase, which we leave to the student.

$\square$

Note that the Mayer–Vietoris sequence works also for reduced homology provided $X_1 \cap X_2 \neq \emptyset$.

8.1. Applications of the Mayer–Vietoris Sequence.

PROPOSITION 6.32. *For the 2-torus T^2, we have*

$$\begin{aligned} H_i(T^2) &= \mathbf{Z} \qquad i = 0 \\ &= \mathbf{Z} \oplus \mathbf{Z} \qquad i = 1 \\ &= \mathbf{Z} \qquad i = 2 \\ &= 0 \qquad \textit{otherwise.} \end{aligned}$$

PROOF. We give two arguments. (i) Identify the torus as the unit square with opposite edges identified as below. Define open sets U and V as indicated in the diagram. Then $T = U \cup V$, U and V each have a circle as a deformation retract, and $U \cap V$ has two connected components, each of which has a circle as a deformation retract. Hence, the Mayer–Vietoris sequence gives

$$H_2(U) \oplus H_2(V) = 0 \to H_2(T) \to H_1(U \cap V) = \mathbf{Z} \oplus \mathbf{Z} \to H_1(U) \oplus H_1(V) = \mathbf{Z} \oplus \mathbf{Z} \to$$
$$H_1(T) \to \tilde{H}_0(U \cap V) = \mathbf{Z} \to \tilde{H}_0(U) \oplus \tilde{H}_0(V) = 0.$$

Hence, we need to identify the middle homomorphism $\mathbf{Z}^2 \to \mathbf{Z}^2$. In the map $H_1(U \cap V) \to H_1(U)$, each basis element b_i of $H_1(U \cap V)$ maps to a generator b' of $H_1(U)$. Similarly, under $H_1(U \cap V) \to H_1(V)$, each b_i maps to a generator b'' of $H_1(V)$. Since $i_* = i_{1*} \oplus -i_{2*}$, the middle homomorphism may be described by the two by two integral matrix

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Writing elements of $\mathbf{Z}^2$ as column vectors, we see that a basis for the kernel of this homomorphism is

$$b_1 - b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and it is free of rank one. It follows from this that $H_2(T) = \mathbf{Z}$. Similarly, since the the image of the homomorphism is spanned by the columns of the matrix, $b' - b''$ is also a basis for the image. Since $\mathbf{Z}^2 = \mathbf{Z}(b_1 - b_2) \oplus \mathbf{Z}b_2$, it follows that the cokernel of the homomorphism is also free of rank one. Hence, we have a short exact sequence

$$0 \to \mathbf{Z} \to H_1(T) \to \mathbf{Z} \to 0$$

which necessarily splits. Hence, $H_1(T) \cong \mathbf{Z} \oplus \mathbf{Z}$ as claimed.

(ii) Argue instead as we did when applying the Seifert-VanKampen Theorem to calculate the fundamental group of T. Choose U to be the square with its boundary eliminated, and V to be the punctured torus as indicated below. Then U is contractible, V has a wedge of two circles as a deformation retract, and $U \cap V$ has a circle as a deformation retract. By a simple argument using the Mayer–Vietoris sequence, it is possible to see that the wedge of two circles has trivial reduced homology in all dimensions except one, and $H_1 = \mathbf{Z} \oplus \mathbf{Z}$. Each loop of the wedge provides a basis element. (H_1 may also be calculated by using the isomorphism with $\pi_1/[\pi_1, \pi_1]$. (See the Exercises.) Now apply the Mayer-Vietoris sequence to get

$$0 \to H_2(T) \to H_1(U \cap V) = \mathbf{Z} \to H_1(U) \oplus H_1(V) = 0 \oplus (\mathbf{Z} \oplus \mathbf{Z}) \to H_1(T) \to 0.$$

However, the diagram below shows that a generator of $H_1(U \cap V)$ maps to zero in $H_1(V)$, so we obtain the desired results.

□

Note that in the homomorphism $j_* : H_1(U) \oplus H_1(V) \to H_1(T)$ a generator of either factor goes onto the summand $\mathbf{Z}$ on the left. In either case, it is not hard to see that this can be identified with a singular 1-simplex consisting of a loop going around the torus in one direction. Clearly, the other loop should provide the other generator, but this is not clear from the above argument. However, if we use the isomorphism

$$\pi_1(T)/[\pi_1(T), \pi_1(T)] \cong H_1(T)$$

discussed previously, it is clear that we can identify two loops for the torus as a basis for $H_1(T)$ since we already know they form a basis for the fundamental group, which in this case is free abelian of rank two. This is clearly a quicker and more effective way to compute H_1, but we need the Mayer–Vietoris sequence to compute H_2.

9. Some Important Applications

In this section, we shall prove certain important classical theorems. The Jordan Curve Theorem asserts that any simple closed curve C in $\mathbf{R}^2$ divides the plane into two regions each of which has C as its boundary. We shall prove a generalization of this to higher dimensions.

First we need some preliminaries.

A closed r-cell in S^n is any subspace e^r which is homemorphic to the standard r-cell $I^r = I \times I \times \cdots \times I$. Note that since S^n is not contractible, a closed r-cell cannot be all of S^n.

THEOREM 6.33. *Let e^r be a closed r-cell in S^n, $n, r \geq 0$. Then $\tilde{H}_q(S^n - e^r) = 0$ for all q.*

Proof. The result is clear for $n = 0$. Suppose $n > 0$. We shall proceed by induction on r. The result is true for $r = 0$, since a 0-cell is a point and in that case $S^n - e^r$ is contractible. Suppose it has been proven for $0, 1, \ldots, r-1$. We may decompose

$$I^r = (I^{r-1} \times [0, 1/2]) \cup (I^{r-1} \times [1/2, 1])$$

and corresponding to this we get a decomposition of $e^r = e' \cup e''$. compact sets, they are closed in S^n, so $S^n - e'$ and $S^n - e''$ are open sets in S^n. Their intersection is $S^n - e^r$. Their union on the other hand is $S^n - e' \cap e''$, but $e' \cap e''$ is homemorphic to

$$I^{r-1} \times [0, 1/2] \cap I^{r-1} \times [1/2, 1] = I^{r-1}$$

so it is an $(r-1)$-cell. Hence, by induction $\tilde{H}_q(S^n - e' \cap e'') = 0$ for all q. From the Mayer–Vietoris sequence this implies that

$$\tilde{H}_q(S^n - e^r) \cong \tilde{H}_q(S^n - e') \oplus \tilde{H}_1(S^n - e'')$$

for all q. Suppose for some q that the left hand side is non-trivial. Then one of the groups on the right is non-trivial; say $H_q(S^n - e') \neq 0$. However, e' is homemomorphic to $I^{r-1} \times [0, 1/2] \simeq I^r$, so it is also a closed r-cell. Clearly, we may iterate this argument to obtain a sequence of closed r-cells $E_1 = e^r \supseteq E_2 \supseteq E_3 \supseteq \ldots$ with $e^{r-1} = \cap_i E_i$ a closed $(r-1)$-cell. Also, for each i, $\tilde{H}_q(S^n - E_i) \to \tilde{H}_1(S^n - E_{i+1})$ is a non-trivial monomorphism onto a direct summand.

Lemma 6.34. *Let $X_1 \subseteq X_2 \subseteq X_3 \ldots$ be an ascending chain of spaces each of which is open in the union $X = \cup_i X_i$.*

(a) Suppose $\gamma \in \tilde{H}_q(X_i)$ maps to zero in $\tilde{H}_q(X)$. Then there is a $j > i$ such that γ maps to zero in $\tilde{H}_q(X_j)$.

(b) For each $\rho \in \tilde{H}_q(X)$, there exists an i such that ρ is the image of some $\gamma \in \tilde{H}_q(X_i)$.

Note that part (a) of Lemma 6.34 proves the theorem. For, we may take $X_i = S^n - E_i$. Then, if $\gamma \in H_q(X_1)$ is non-trival, it must map to zero in $H_q(X) = H_q(S^n - e^{r-1}) = 0$. Hence, it must map to zero in some $\tilde{H}_q(X_j)$ with $j > 1$, but that can't happen by the construction. □

Proof of Lemma 6.34. We shall prove the Lemma for ordinary homology. The case of reduced homology for $q = 0$ is similar, and we leave it for the student.

(a) Let c be a cycle representing $\gamma \in H_q(X_i)$. The support c is the union of a finite number of supports of singular simplices, so it is compact. Let $c = \partial x$ for some singular chain x in X. The support of x is also compact. Let A be the union of the supports of c and x. Since A is compact, and since the X_j form an open covering, it follows that A

is covered by finitely many X_j. Since the subspaces form an ascending chain, it follows that there is a j such that $A \subseteq X_j$. It is clear that the homology class of c is zero in $H_q(X_j)$, and we may certainly take $j > i$ if we choose.

(b) Let c be a cycle representing $\rho \in H_q(X)$. As above, the support of c is compact and contained in some X_i. □

THEOREM 6.35. *Let S be a proper subspace of S^n which is homeomorphic to S^r with $n > 0$, $r \geq 0$. Then*

$$\tilde{H}_q(S^n - S) = \begin{cases} \mathbf{Z} & \textit{if } q = n - r - 1 \\ 0 & \textit{otherwise.} \end{cases}$$

This Theorem has several interesting consequences.

COROLLARY 6.36. *Suppose S is a proper subspace of S^n which is homeomorphic to S^r with $n > 0$, $r \geq 0$. Then we must have $r < n$. If $r \neq n - 1$, then $S^n - S$ is path connected. In particular if K is a knot in S^3, then $S^3 - K$ is path connected. If $r = n - 1$, then $S^n - S$ has two components.*

PROOF OF COROLLARY 6.36. If $r \geq n$, then we would have a nonzero homology group in a negative dimension, which is impossible. Since S is compact, it is closed in S^n, so its complement is open and is necessarily locally path connected. Hence, the path components are the same as the components. Moreover, the rank of $\tilde{H}_0$ is one less than the number of path components, and $\tilde{H}_0(S^n - S) = 0$ unless $r = n - 1$ in which case it is $\mathbf{Z}$. □

PROOF OF THEOREM 6.35. We proceed by induction on r. For $r = 0$, S consists of two points, and $S^n - S$ is homeomorphic to $\mathbf{R}^n$ less a point, so it has S^{n-1} as a deformation retract. Since

$$\begin{aligned} \tilde{H}_q(S^{n-1}) &= 0 \qquad q \neq n - 1 = n - r - 1 \\ \tilde{H}_{n-1}(S^{n-1}) &= \mathbf{Z}, \end{aligned}$$

the result follows for $r = 0$.

Suppose the theorem has been proved for $0, 1, \ldots, r-1$. Decompose $S^r = D^r_+ \cup D^r_-$ where $D^r_+ = \{x \in S^r \mid x_r \geq 0\}$ and $D^r_- = \{x \in S^r \mid x_r \leq 0\}$. Let $S = e' \cup e''$ be the corresponding decomposition of S. Note that e' and e'' are closed r-cells and $S' = e' \cap e''$ is homeomorphic to S^{r-1}. Now apply the reduced Mayer–Vietoris sequence to the open covering

$$S^n - S' = S^n - e' \cap e'' = (S^n - e') \cup (S^n - e''),$$

where $(S^n - e') \cap (S^n - e'') = S^n - e' \cup e'' = S^n - S$. (Note that this requires $S^n - S \neq \emptyset$.) Since $S^n - e'$ and $S^n - e''$ are acyclic, i.e., they have trivial reduced homology in every dimension, it follows that

$$\tilde{H}_{q+1}(S^n - S') \cong \tilde{H}_q(S^n - S)$$

for every q. However, by induction, the left hand side is zero unless $q+1 = n-(r-1)-1 = n-r$ in which case it is $\mathbf{Z}$. But $q+1 = n-r$ if and only if $q = n-r-1$, so we get what we need for the right hand side. □

We have now done most of the work for proving the following theorem.

THEOREM 6.37 (Jordan-Brouwer Separation Theorem). *Let S be a subspace of S^n homemorphic to S^{n-1} for $n > 0$. Then $S^n - S$ has two connected components each of which has S as its boundary.*

PROOF. We already know that $S^n - S$ has two components U and V. All that remains is to show that both have S as boundary. Consider $\overline{U} - U$. Since U and V are disjoint open sets, no point of V is in $\overline{U} - U$. Hence, $\overline{U} - U \subseteq S$.

Conversely, let $x \in S$. Let W be an open neighborhood of x. We may choose a decomposition of $S \simeq S^{n-1}$ into two closed $(n-1)$-cells e', e'' with a common boundary and we may assume one of these $e' \subset W$. Since $S^n - e''$ is path connected ($\tilde{H}_0(S^n - e'') = 0$), we may choose a path α in S^n connecting some point of U to some point in V which does not pass thru e''. Let s be the least upper bound of all t such that $\alpha([0,t]) \subseteq U$. Certainly, $\alpha(s) \in \overline{U}$, so it is not in V, but since U is open $\alpha(s)$ is also not in U. Hence, $\alpha(s) \in S$. Since it is not in e'', $\alpha(s) \in e' \subseteq W$. Thus, it follows that every open neighborhood W of x contains a point of $\overline{U}$. Hence, x is in the closure of $\overline{U}$, i.e., it is in $\overline{U}$. This shows that $S \subseteq \overline{U}$, but since $S \cap U = \emptyset$, we conclude finally that $S \subseteq \overline{U} - U$. □

COROLLARY 6.38 (Jordan-Brouwer). *Let $n \geq 2$ and let S be a subspace of $\mathbf{R}^n$ which is homeomorphic to S^{n-1}. Then S separates $\mathbf{R}^n$ into two components, one of which is bounded and one of which is unbounded.*

PROOF. Exercise. □

You should think about what happens for $n = 1$.

In $\mathbf{R}^2$, by a theorem of Schoenflies, the bounded component is homemorphic to an open disk. However, this result does not extend to higher dimensions unless one makes extra assumptions about the imbedding of S^{n-1} in $\mathbf{R}^n$. There is an example, called the *Alexander horned sphere*, of a homemomorph of S^2 in $\mathbf{R}^3$ where the components of the complement are not even simply connected. It is constructed as follows.

Start with a two torus, and cut out a cylindrical section as indicated in the above diagram to the left. Put a cap on each end, so the result is homeomorphic to S^2. On each cap, erect a handle, cut out a cylindrical section, cap it, etc. Continue this process indefinitely. At any finite stage, you still have a space homeomorphic to S^2. Now take the union of all these surfaces and all limit points of that set. (In fact, the set of limit points forms a Cantor set.) It is possible to see that the resulting subpace of $\mathbf{R}^3$ is also homemorphic to S^2. Next modify the process as follows. When adding the pair of handles to each opposing pair of caps, arrange for them to link as indicated in the above diagram to the right. Otherwise, do the same as before. It is intuitively clear that the outer complement of the resulting subspace is not simply connected, although it might not be so easy to prove. Also, the space is still homeomorphic to S^2, since the linking should not affect that. Now modify this process at any stage (more than once if desired), as on the right of the above diagram, by deforming two of these cylinders and linking them as indicated. This deformation does not change the fact that the space is homeomorphic to a sphere, but it is clear that the complement is not simply connected.

Theorem 6.39 (Invariance of Domain). *Let U be an open set in $\mathbf{R}^n$ (alternately S^n) and let $h : U \to \mathbf{R}^n\,(S^n)$ be a one-to-one continuous map. Then $h(U)$ is open, and h provides a homeomorphism from U to $h(U)$.*

This theorem is important because it insures that various pathological situations cannot arise. For example, in the definition of a manifold, we required that every point have a neighborhood homeomophic to an open ball in $\mathbf{R}^n$ for some fixed n. Suppose we allowed different n for different points. If two such neighborhoods intersect, the theorem on invariance of domain may be used to show that the 'n' for those neighborhoods must be the same. Hence, the worst that could happen would be that the dimension n would be different for different components of the space.

Note that 'invariance of domain' is not necessarily true for spaces other than $\mathbf{R}^n$ or S^n. For example, take $U = [0, 1/2)$ as an open subset of $[0, 1]$ and map it into $[0, 1]$ by $t \mapsto t + 1/2$. The image $[1/2, 1)$ is not open in $[0, 1]$.

PROOF. It suffices to consider the alternate case for S^n. (Why?) Let $x \in U$. Choose a closed ball D centered at x and contained in U. Since D is compact, $h(D)$ is in fact homemomorphic to D, so it is a closed n-cell. It follows that $S^n - h(D)$ is connected. Hence, $S^n - h(\partial D) = (S^n - h(D)) \cup h(D^\circ)$ is a decomposition of $S^n - h(\partial D)$ into two disjoint connected sets. Since the Jordan-Brouwer Theorem tells us that $S^n - h(\partial D)$ has precisely two components, both or which are open, it follows that one of these sets is $h(D^\circ)$, so that set is an open neighborhood of $h(x)$. It follows from this that $h(U)$ is open. $\square$

CHAPTER 7

Simplicial Complexes

1. Simplicial Complexes

Singular homology is defined for arbitrary spaces, but as we have seen it may be quite hard to calculate. Also, if the spaces are bizarre enough, the singular homology groups may not behave quite as one expects. For example, there are subspaces of $\mathbf{R}^n$ which have non-zero singular homology groups in every dimension. We now want to devote attention to nicely behaved spaces and to derive alternate approaches to homology which give reasonable approaches. We start with the notion of a simplicial complex.

We need to clarify some issues which were previously ignored. The notation $[p_0, p_1, \dots, p_n]$ for an affine n-simplex in $\mathbf{R}^N$, assumes an implicit order for the vertices. This order played an important role in the development of the theory. Certainly, the order is essential if we interpret the notation as standing for the affine map $\Delta^n \to \mathbf{R}^N$ defined by $\mathbf{e}_i \mapsto p_i$. However, even without that, the order is used implicitly when numbering the $(n-1)$-dimensional faces of the simplex. Earlier, the notation took care of itself, so we didn't always have to distinguish between an affine simplex as a convext subset of a Euclidean space and an ordered affine simplex in which an order is specified for the vertices. In what follows, we shall have to be more careful. We shall use the term 'affine simplex' to mean the point set and 'ordered affine simplex' to mean the set together with an order for the vertices, or what is the same thing, the affine map $\Delta^n \to \mathbf{R}^N$.

In what follows, by the term 'face of an affine simplex', we shall mean any affine simplex spanned by a non-empty subset of its vertices, rather than specifically one of the faces in its boundary.

A simplicial complex K consists of a collection of affine simplices α in some $\mathbf{R}^n$ which satisfy the following conditions.

(1) Any face of a simplex in K is in K.

(2) Two simplices in K either are disjoint or intersect in a common face.

Clearly, any affine simplex together with all its faces provide a collection of simplices which are a simplicial complex. Other examples are indicated in the diagram below.

We shall consider only finite simplicial complexes, i.e., ones made up of a finite number of simplices. Although this assumption is implicit, we shall usually state it explicitly in important theorems and propositions, but on occasion we will forget. A simplicial complex K will be called n-dimensional if n is the largest dimension of any simplex in K.

If K is a simplicial complex, we denote by $|K|$ the space which is the union of all the simplices in K. (Note that if K were not finite, there might be some ambiguity about what topology to use on $|K|$. For example, in $\mathbf{R}^\infty$ one can consider the convex set spanned by the standard basis vectors $e_i, i = 0, 1, \ldots$. In the finite case, $|K|$ will be a compact subset of some finite dimensional $\mathbf{R}^N$.)

A space X which is homemeomorphic to $|K|$ for some finite simplicial complex K will be called a *polyhedron* and K will be called a *triangulation* of X. Note that a space can have more than one triangulation. For example, the 2-faces of a tetrahedron form a triangulation of S^2, but by dividing each face of a cube into two triangles, we obtain a different triangulation.

More generally, if K is a simplicial complex, then the process of barycentric subdivision defined previously, when applied to the simplices in K produces another simplicial complex we shall denote $Sd(K)$. (You should review the definitions used in barycentric subdivision to see that the subdivision of a simplicial complex is again a simplicial complex.) Clearly, $|Sd(K)| = |K|$, but they are different simplical complexes.

If K is a simplicial complex, let $vert(K)$ denote its set of vertices or 0-simplices.

If K and L are simplicial complexes, we say that L is a subcomplex of K if $L \subseteq K$ and $vert(L) \subseteq vert(L)$. Note that a subcomplex can have exactly the same vertices as the complex without being the same. For example. consider the boundary of a simplex.

If K and L are simplicial complexes, a morphism $\phi : K \to L$ is a function $\phi : vert(K) \to vert(L)$ such that if σ is a simplex of K spanned by the affinely independent set $\{p_0, \ldots, p_m\}$, then the distinct elements of the set elements in the list $\phi(p_0), \ldots, \phi(p_m)$ form an affinely independent set spanning a simplex $\phi(\sigma)$ in L. Note that ϕ need not be one-to-one, and $\phi(\sigma)$ could have lower dimension that σ.

The collection of simplicial complexes and morphisms of such forms a category we denote $\mathcal{K}$.

Given a morphism $\phi : K \to L$ of simplicial complexes, for each simplex, $p_i \mapsto \phi(p_i)$ defines a unique affine map of the affine simplex σ to $\phi(\sigma)$. This induces a piecewise-affine map of spaces $|\phi| : |K| \to |L|$. It is not hard to see that $| - |$ is a functor from the category $\mathcal{K}$ of simplicial complexes to the category Top of toplogical spaces.

Generally, given a space X, it may have no triangulation. (There are even n-manifolds for large n with that property.) Even for simplicial complexes, one can have a map $f : |K| \to |K|$, which is not the realization of a simplicial morphism. For example, S^1 may be triangulated, say as the edges of a triangle, but we can define a degree two map $S^1 \to S^1$ which won't take all vertices to vertices.

However, it turns out that, given a map f of polyhedra, *after sufficiently many subdivisions*, we can find a simplicial morphism ϕ such that $|\phi|$ is homotopic to f.

To prove this we must first introduce some notation. If σ is an affine p-simplex, let $\dot{\sigma}$ denote the boundary of σ and let $\mathring{\sigma} = \sigma - \dot{\sigma}$ denote the interior of σ *except* if $p = 0$ in which case let $\mathring{\sigma} = \sigma$.

Let K be a simplicial complex, and let p be a vertex of K.

$$St(p) = \bigcup_{p \in vert(\sigma), \sigma \in K} \mathring{\sigma}$$

is an open subset of $|K|$ called the *star* of p.

Let K and L be simplicial complexes, and let $f : |K| \to |L|$ be a map. A simplicial morphism $\phi : K \to L$ is called a *simplicial approximation* if for every vertex p of K we have $f(St(p)) \subseteq St(\phi(p))$.

Note first that if ϕ is a simplicial approximation for f, then for each $x \in |K|$, $|\phi|(x)$ and $f(x)$ are in a common simplex of L. For, choose a simplex σ of minimal dimension in K, such that $x \in \mathring{\sigma}$. Because simplices interset only in faces, σ is unique. We claim that $\phi(\sigma)$ is *a face* of some simplex in L containing $f(x)$, (Of course, $|\phi(x)| \in \phi(\sigma)$, so that suffices.) The claim follows from the Lemma below since for each of the vertices p_i of σ we have $f(x) \in St(\phi(p_i))$.

LEMMA 7.1. *Let L be a simplicial complex, and let $\{q_0, \ldots, q_r\}$ be a set of distinct vertices of L. If a point $y \in |L|$ lies in $\cap_{i=0}^{r} St(q_i)$, then there is a simplex τ in L such that $y \in \tau$, and $||[q_0, \ldots, q_r]||$ is a face of τ.*

PROOF. Since $y \in St(q_i)$, there is a simplex τ_i in L with q_i a vertex and such that $y \in \mathring{\tau}_i$. However, as above, y belongs to $\mathring{\tau}$ for precisely one simplex τ in L. Hence, all the τ_i are the same simplex τ, and each q_i is a vertex of τ. Hence, the q_i span a face of τ as claimed. □

It follows from the above discussion that if ϕ is a simplicial approximation to f, then $|\phi|$ is homotopic to f. For, since for every x, $f(x)$ and $|\phi|(x)$ always belong to a common convex subset of a Euclidean space, the formula

$$F(x,t) = tf(x) + (1-t)|\phi|(x) \qquad x \in |K|, 0 \leq t \leq 1$$

makes sense and defines a homotopy from one to the other.

THEOREM 7.2. *K and L be finite simplicial complexes, and let $f : |K| \to |L|$ be a map. Then there exists a simplicial approximation $\phi : Sd^k K \to L$ of f for some $k \geq 0$.*

PROOF. The sets $f^{-1}(St(q))$ for q a vertex of L form an open covering of $|K|$. Since K is finite, $|K|$ is compact. Hence, by the Lebesgue Covering Lemma, there is a $\delta > 0$ such that every subset of $|K|$ of diameter less that δ is carried by f into a star of some vertex of L. Choose k such that the mesh of $Sd^k\sigma$ is less that $\delta/2$ for each $\sigma \in K$. It follows easily that each $St(p)$ for p a vertex of $Sd^k K$ has diameter less that δ, so $f(St(p)) \subseteq St(q)$ for some vertex q in L. Choose such a vertex and set $\phi(p) = q$.

This defines a map $\phi : vert(Sd^k K) \to vert(L)$. To complete the proof, we need to show that $\phi(\sigma)$ is a simplex in L whenever $\sigma = |[p_0, \dots, p_m]|$ is a simplex in $Sd^k K$. Let $x \in \sigma$. Then $x \in \cap_i St(p_i)$. Hence,

$$f(x) \in f(\cap_i St(p_i)) \subseteq \cap_i f(St(p_i)).$$

By the above Lemma, the distinct elements in the list $\phi(p_0), \dots, \phi(p_m)$ span a simplex in L which is a face of a simplex in L containing $f(x)$. It follows that $\phi(\sigma)$ is a simplex in L. $\square$

For any simplicial complex K, the collection of all simplices σ in K of dimension less than or equal to r is again a simplicial complex called the *r-skeleton* of K. (Check this for yourself.) We shall denote it $K^{(r)}$. Note $K^{(n)} = K$ if K is n-dimensional.

Suppose $\phi : K \to L$ is a simplicial morphism. If σ has dimension $\le r$, then certainly $\phi(\sigma)$ has dimension $\le r$. It follows easily that ϕ induces a simplicial morphism $\phi_r : K^{(r)} \to L^{(r)}$. Clearly,

$$|\phi|(|K^{(r)}|) \subseteq |L^{(r)}|.$$

THEOREM 7.3. *Any map $f : S^r \to S^n$ with $0 < r < n$ is null homotopic, i.e., homotopic to a trivial map.*

In the language of homotopy groups, this asserts $\pi_r(S^n) = 0$ for $0 < r < n$. This generalizes the fact that S^n is simply connected for $n \ge 2$.

PROOF. The difficulty is that f might be onto. Otherwise, the image of f is contained in a subspace of S^n homeomorphic to $\mathbf{R}^n$ which is contractible. S^n is homeomorphic to $|L|$ where L is the simplicial complex obtained from the boundary of Δ^{n+1}. Similarly, S^r is homeomorphic to $|K|$ where K is the boundary of Δ^{r+1}. Choose a k for which there is a simplicial approximation $\phi : Sd^k K \to L$ to f. $|\phi|(|K|) \subseteq |L^{(r)}|$, so $|\phi|$ is not onto. Hence, $|\phi|$ is null homotopic. Since $f \sim |\phi|$, it follows that f is also null homotopic. $\square$

2. Abstract Simplicial Complexes

Let K be a finite simplicial complex. We think of K as a subset of a Euclidean space decomposed into simplices. However, all the relevant information about K is contained in knowledge of its set of vertices and which subsets of that set span simplices. This suggests the following more abstract approach.

An *abstract simplicial complex* K consists of a set V (called vertices) and a collection of distinguished non-empty finite subsets of V (called simplices) such that

(a) every singleton subset $\{p\}$ of V is a simplex;

(b) every non-empty subset of a simplex is also a simplex (called a face of the simplex).

Note that it follows that the intersection of two simplices is either empty or a face of both simplices.

A morphism of abstract simplicial complexes is a function from the set of vertices of the first to the set of vertices of the second which takes simplices to simplices. It is not hard to check that the collection of abstract simplicial complexes and morphisms of such forms a category.

As noted above, a simplicial complex K defines an abstract simplicial complex K^a in the obvious way. Similarly, a simplicial morphism $\phi : K \to L$ of simplicial complexes defines a morphism $\phi^a : K^a \to L^a$ in the obvious way. It is not hard to check that $(-)^a$ is a functor from the category of simplicial complexes to the category of abstract simplicial complexes.

There is a (non-functorial) way to go in the opposite direction. Let K be a finite abstract simplicial complex. If K' is a simplicial complex with $K'^a \cong K$,i.e., there are a pair a asbtract simplicial morphisms relating $(K')^a$ and K which are inverses of one another. Then $|K'|$ is called a *geometric realization* of K. There is a standard way we may always construct such a geometric realization. Let $n+1$ be the number of vertices in K. Consider the simplicial complex L^n obtained from the standard simplex Δ^n and all its faces. Choose some one-to-one correspondence between the vertices of K and the vertices of Δ^n. This amounts to an ordering

$$v_0, v_1, \ldots, v_n$$

of the vertices of K. Let K^g be the subcomplex of L^n consisting of those simplices

$$\sigma = |[e_{i_0}, \ldots, e_{i_r}]|$$

such that $\{v_{i_0}, \ldots, v_{i_r}\}$ is a simplex of K. It is clear that $|K^g|$ is a geometric realization of K.

It is not hard to see that any two geometric realizations of an abstract simplicial complex are homeomorphic. For suppose that $K_1^a \cong K_2^a$. Then we can define simplicial morphisms $\phi : K_1 \to K_2$ and $\psi : K_2 \to K_1$ in the obvious manner such that the compositions in both directions are the identity simplicial morphisms of K_1 and K_2 respectively. It follows that $|\phi| : |K_1| \to |K_2|$ and $|\psi| : |K_2| \to |K_2|$ are inverse maps of spaces, so the spaces are homeomorphic.

3. Homology of Simplicial Complexes

There are several possible ways to define homology for a simplicial complex. The difficulty is deciding which signs to attach to the faces of a simplex in the formula for the boundary. This requires some way to specify an orientation for each simplex. Fortunately, any reasonable method will result in homology groups isomorphic to the singular homology groups of the support of the simplicial complex, so it doesn't matter which we choose.

Let K be a simplicial complex (or an abstract simplicial complex). The naive thing to do would be to define $C_*(K)$ as the free abelian group with the set of simplices as basis. In fact this is pretty much what we will actually do. Unfortunately, if we just specify a simplex as by its set of vertices, we have no way to determine the signs for the faces in the formula for its boundary. To deal with this issue we must introduce some notion of *orientation*. Here is one approach, which is a little indirect, but has some conceptual advantages.

Let $O_n(K)$ be the free abelian group with basis the set of all symbols of the form $[p_0, p_1, \dots, p_n]$ where $\{p_0, p_1, \dots, p_n\}$ is the set of vertices of some simplex σ in K, but *we allow possible repetitions in the list*. Call such symbols *abstract ordered n-simplices*. Note that there are $(n+1)!$ abstract ordered n-simplices elements associated with each n simplex σ in K, but there are also many additional *degenerate* degenerate abstract ordered m-simplices with $m > n$. We can define a boundary homomorphism $\partial_n : O_n(K) \to O_{n-1}(K)$ in the usual way with

$$\partial_n[p_0, \dots, p_n] = \sum_{i=0}^{n} (-1)^i [p_0, \dots, \hat{p}_i, \dots, p_n]$$

for $n > 0$. As before, $\partial_{n-1} \circ \partial_n = 0$.

In $O_n(K)$, let $T_n(K)$ be the subgroup generated by all elements of the form

$$[p_0, \dots, p_n] \quad \text{if there is a repetition of vertices,}$$

and

$$[p_0, p_1, \dots, p_n] - s(\pi)[q_0, q_1, \dots, q_n] \qquad \text{if there is no repetition of vertices,}$$

where $q_i = p_{\pi(i)}, i = 0, \dots, n$ for some permutation π of degree $n+1$, and where $s(\pi) = \pm 1$ is the sign of that permutation. It is not hard to check that $\partial_n(T_n) \subseteq T_{n-1}$. (See the Exercises)

Now let $C_n(K) = O_n(K)/T_n(K)$ and let $\partial_n : C_n(K) \to C_{n-1}(K)$ be the induced homomorphism, and again note that $\partial_{n-1} \circ \partial_n = 0$. Hence, we can define the group $Z_n(K) = \operatorname{Ker} \partial_n$ of cycles, the group

$B_n(K) = \operatorname{Im}\partial_{n+1}$ of boundaries, and the homology group $H_n(K) = Z_n(K)/B_n(K)$.

Let $\langle p_0, \dots, p_n\rangle$ denote the element of $C_n(K)$ represented by $[p_0, \dots, p_n]$. Then, if there is no repetition in the list, and we permute the vertices in the list, we either obtain the same element or its negative depending on whether the permutation is even or odd. ($\langle p_0, \dots, p_n\rangle$ is zero if there is a repetition in the list, so it seems a bit superfluous to bother with the degenerate abstract ordered simplices at all, but they will yield some technical advantages later.) It follows that we may form a basis for $C_n(K)$ by choosing for each n-simplex in K, one of these two possible representatives which are the same except for sign. This may be done explicitly as follows. Choose an ordering for the vertices of K with the property that the vertices of any simplex σ in K are linearly ordered

$$p_0 < p_1 < \cdots < p_n.$$

Then, the element $\langle p_0, \dots, p_n\rangle$ of $C_n(K)$ represented by the abstract ordered simplex $[p_0, \dots, p_n]$ is the basis element corresponding to σ. Thus, the ordering of K yields a preferred orientation for each simplex, but different orderings may produce the same set of preferred orientations. Generally, if we were to choose some other acceptable ordering of the vertices of K, we would get another basis differing from the first in that some of the basis elements might be negatives of those determined by the first ordering. To make calculations, one would pick some ordering of the vertices. Then each basis element of $C_n(K)$ may be identified with the appropriate ordered simplex and we just use the usual formula for its boundary.

The simplicial homology groups defined this way have some advantages over the singular homology groups of $|K|$ in the case of a finite simplicial complex K. For, it is clear that the groups $C_n(K), Z_n(K), B_n(K)$, and $H_n(K)$ are all finitely generated abelian groups. In addition, if K is n-dimensional, it is clear that $H_r(K) = 0$ for $r > n$.

The most important disadvantage of simplicial homology is that it appears to depend on the complex rather than just the space $|K|$. However, we shall see below that simplicial homology gives the same result as singular homology for polyhedra.

Let $\phi : K \to L$ be a simplicial map. It is not hard to check that ϕ induces a chain morphism $O_*(K) \to O_*(L)$ taking $T_*(K)$ into $T_*(L)$. Hence, it induces a chain map $\phi_\sharp : C_*(K) \to C_*(L)$ which may be described on basis elements by

$$\phi_\sharp\langle p_0, \dots, p_n\rangle = \langle \phi(p_0) \dots, \phi(p_n)\rangle.$$

Here, according to our conventions the vertices on the left are in the proper order for K, but on the right we might have to introduce a sign in order to get the vertices in the proper order for L. If there is a repetition of vertices on the right, our conventions tell us the symbol stands for zero. This in turn induces homomorphisms $\phi_n : H_n(K) \to H_n(L)$. It is not hard to check that simplicial homology defined in this way is a functor on the category of simplicial complexes and simplicial morphisms.

One may also define *reduced simpicial homology* much as in the singular theory. Define $C_{-1}(K) = \mathbf{Z}$ and define a homomorphism $\tilde{\partial}_0 : C_0(K) \to C_{-1}(K)$ by $\sum_i n_i p_i \mapsto \sum_i n_i$. Then let $\tilde{H}_0(K) = \operatorname{Ker} \tilde{\partial}_0 / \operatorname{Im} \partial_1$. This is also a functor as above.

Call a simplicial complex simplicially connected if given any two vertices, there is a sequence of intermediate vertices such that successive pairs of vertices are the 0-faces of 1-simplices in the complex. Using this notion, one could define simplicial components of a simplicial complex. It is not hard to see that for a finite complex, these notions conicide with the notions of path connectedness and path components for the support of the somplex. It is also easy to see that $H_0(K)$ is the free group with the set of components as basis, and analgously for $\tilde{H}_0(K)$.

Relative homology is also defined as in the singular case. Let K be a simplicial complex, appropriately ordered in some way, and let L be a subcomplex with the inherited order. Then $C_*(L)$ may be identified with a subcomplex of the chain complex $C_*(K)$ and we may define the relative chain complex $C_*(K, L) = C_*(K)/C_*(L)$, and $H_*(K, L)$ is defined to be its homology. It follows from basic homological algebra that we have connecting homomorphisms and a long exact sequence

$$\cdots \to H_n(L) \to H_n(K) \to H_n(K, L) \to H_{n-1}(L) \to \dots$$

and similarly for reduced homology. Moreover, the connecting homomorphisms are natural with respect to simplicial morphisms of pairs (K, L) of simplical complexes. The proofs of these facts parallel what we did before in singular theory and we shall omit them here.

Note that everything we did above could just as well have been done for finite abstract simplicial complexes.

3.1. An Example. Use the diagram below to specify an abstract simplicial complex with geometric realization homeomorphic to a 2-torus. Note that an actual geometric simplicial complex would in fact have to be imbedded in a higher dimensional space than $\mathbf{R}^2$, say in $\mathbf{R}^3$. Call such a simplicial complex K.

Note that K has 9 vertices (0-simplices), 27 1-simplices, and 18 2-simplices. The numbering of the vertices indicates an acceptable ordering, and the arrows show induced orientations for the 1-simplices. For the 2-simplices, the arrows indicate the simplex should be counted with $\pm$ according to whether they are consistent with the ordering or not.

$H_0(K) = \mathbf{Z}$ since the complex is connected. We next use a mixture of theory and explicit calculation to see that $H_1(K) = \mathbf{Z} \oplus \mathbf{Z}$ and $H_2(K) = \mathbf{Z}$ as we would expect from our calculations in the singular case. Let L be the subcomplex of K consisting of the 6 1-simplices on the edges and their vertices. We first determine the homology groups of L. (Of course, $|L|$ is homemorphic to a wedge $S^1 \vee S^1$.) $H_n(L) = 0$ for $n > 1$ and $\tilde{H}_0(L) = 0$. To compute $H_1(L)$ consider the two cycles which are the sums of the 1-simplices on either horizontal or vertical edges. Since $B_1(L) = 0$, these form an independent pair. However, it is easy to see that any 1-chain is a one cycle if and only if it is a linear combination of these two 1-cycles. For, if two adjacent 1-simplices abut in any vertex but that numbered 1, they must have the same coefficient or else some multiple of the common vertex would be non-zero in the boundary. It follows that $H_1(L) = \mathbf{Z} \oplus \mathbf{Z}$.

We next calculate $H_1(K, L)$ and $H_2(K, L)$.

The second is easier. Clearly, the sum of all the 2-simplices (with the appropriate signs as discussed above) represents a 2-cycle in $Z_2(K, L)$. By considering adjacent 2-simplices *inside the square* it is clear that they must have the same coeficient in any 2-cycle in $Z_2(K, L)$ or else the boundary would contain their common edge with a non-zero coefficient. Hence, $H_2(K, L) = \mathbf{Z}$. However, the boundary of the basic 2-cycle in $Z_2(K, L)$ is zero, so the homomorphism $H_2(K, L) \to H_1(L)$ is trivial.

Now cosider $H_1(K, L)$. The diagram below suggests a sequence of reductions to subcomplexes of K containing L. If we start with a 1-cycle modulo $C_1(L)$, and reduce modulo boundaries, we can reduced appropriate 1-simplices to chains in a smaller complex. Any time we have a 1-simplex 'hanging' with a vertex not in L exposed, we can drop it since the boundary in that case would contain some multiple of that vertex. Hence, the original chain could not have been a 1-cycle. At the end of this sequence, we end up in $Z_1(L, L) = 0$. Hence, $H_1(K, L) = 0$.

We may now compute the homology of K from the long exact sequence

$$0 \to H_2(K) \to H_2(K, L) = \mathbf{Z} \xrightarrow{0} H_1(L) = \mathbf{Z} \oplus \mathbf{Z} \to H_1(K) \to H_1(K, L) = 0$$

to obtain

$$H_1(K) = \mathbf{Z} \oplus \mathbf{Z} \tag{1}$$

$$H_2(K) = \mathbf{Z} \tag{2}$$

The argument also shows that the two basic 1-cycles in L map to a basis of $H_1(K)$ and the sum of all the 2-simplices generates $H_2(K)$.

The student should check that these results are consistent with the ranks of $C_n(K), n = 0, 1, 2$. Note that a direct brute force calculation of the homology groups using the formulas for the boundaries would have been quite horrendous. Fortunately, the geometry helps us organize the calculation. Note also that the calculation above could have been done directly in $C_*(K)$ with appropriate modifications. (Try it yourself!) We used the machinery of the long exact sequence because it illuminates the calculation somewhat, but it isn't really necessary.

4. The Relation of Simplicial to Singular Homology

Let K be a finite simplicial complex, and suppose that we choose an acceptable ordering of its vertices. We define a chain morphism $h_\sharp : C_*(K) \to S_*(|K|)$ as follows. For σ an n-simplex in K, let $[p_0, \dots, p_1]$ be the ordered affine simplex corresponding to it for the specified order. Let $h_\sharp(\sigma)$ be the same ordered affine simplex viewed now as a singular simplex (i.e., an affine map $\Delta^n \to |K|$) in $|K|$. Because the boundaries are defined the same way, it is easy to see that $h_\sharp$ is in fact a chain morphism. Hence, it induces homomorphisms $h_n : H_n(K) \to H_n(|K|)$. Using the corresponding reduced complexes, we also get $\tilde{h}_0 : \tilde{H}_0(K) \to \tilde{H}_0(|K|)$ in dimension 0. It is the purpose of this section to show that these homomorphisms are isomorphisms. These isomorphisms appear to depend on the ordering of the vertices of K, but in fact they do not so depend, as we shall see later.

To establish that h_* is an isomorphism, we need to verify some of the 'axioms' we described in singular theory for simplicial theory.

PROPOSITION 7.4. *Let T^n denote the simplicial complex consisting of an affine n-simplex and all its faces. Then $H_q(T^n) = 0$ for $n > 0$, and $H_0(T^n) = \mathbf{Z}$.*

PROOF. We shall prove this by constructing a contracting homotopy for the chain complex $C_*(K)$. Assume a specified ordering $p_0 <$

$p_1 < \cdots < p_n$ for the vertices of the underlying affine n-simplex. Define a homomorphism $t_k : C_k(T^n) \to C_{k+1}(T^n)$ by

$$t_k\langle q_0, \dots, q_k\rangle = \langle p_0, q_0, \dots, q_k\rangle \qquad k \geq 0,$$

where $q_0 < \cdots < q_k$ is the set of vertices (in proper order) of some k-face of the n-simplex if $k \geq 0$ and set $t_k = 0$ otherwise. Note that on the right hand side, $p_0 \leq q_0$ is necessarily true and the result is zero if they are equal. We have for $k > 0$.

$$\partial_{k+1}(\langle p_0, q_0, \dots, q_k\rangle) = \langle q_0, \dots, q_k\rangle - \sum_{i=0}^{k} \langle p_0, q_0, \dots, \hat{q}_i, \dots, q_k\rangle \tag{3}$$

$$= \langle q_0, \dots, q_k\rangle - t_{k-1}(\partial_k\langle q_0, \dots, q_k\rangle. \tag{4}$$

For $k = 0$, the calculation yields

$$\partial_1\langle p_0, q_0\rangle = q_0 - p_0 \tag{5}$$

$$= q_0 - p_0 - t_{-1}(\partial_0 q_0). \tag{6}$$

Let ϵ denote the simplicial morphism of T^n which sends all vertices to p_0. Clearly, $\epsilon_k = 0$ in homology for $k > 0$, and the above formulas show that the homomorphisms t_k form a chain homotopy of $C_*(K)$ from $\epsilon_\sharp$ to $\mathrm{Id}_\sharp$. □

Let K be a simplicial complex, K_1, K_2 subcomplexes. The set of simplices common to both is denoted $K_1 \cap K_2$ and it is easy to see that it is a subcomplex. Similarly, the set of simplices in one or the other is denoted $K_1 \cup K_2$. As in the case of singular homology, we have homomorphisms

$$i_{1*} \oplus -i_{2*} : H_*(K_1 \cap K_2) \to H_*(K_1) \oplus H_*(K_2) \tag{7}$$

$$j_{1*} + j_{2*} : H_*(K_1) \oplus H_*(K_2) \to H_*(K). \tag{8}$$

THEOREM 7.5. *be a finite simplicial complexes, K_1 and K_2 subcomplexes such that $K = K_1 \cap K_2$. Then there are homomorphisms $\partial_n : H_n(K) \to H_{n-1}(K_1 \cap K_2)$ which are natural with respect to triples K, K_1, K_2 ($K = K_1 \cup K_2$) and such that*

$$\cdots \to H_n(K_1 \cap K_2) \to H_n(K_1) \oplus H_n(K_2) \to H_n(K) \to H_{n-1}(K_1 \cap K_2) \to \dots$$

is exact. If $K_1 \cap K_2 \neq \emptyset$, then the corresponding sequence in reduced homology is also exact.

PROOF. We have a short exact sequence of chain complexes

$$0 \to C_*(K_1 \cap K_2) \to C_*(K_1) \oplus C_*(K_2) \to C_*(K_1) + C_*(K_2) = C_*(K) \to 0.$$

We leave the chain morphisms for you to invent, after considering the corresponding morphisms in the singular case. This induces the desired long exact seqeunce. □

There is also a simplicial version of excision. Since we don't have the notion of interior and closure just with the category of simplicial complexes, this 'axiom' must be stated a bit differentely, but it is very easy to prove. (See the Exercises.)

Our strategy in relating simplicial homology to singular homology is to proceed by induction on the number of simplices in K. For zero dimensional simplicial complexes, it is clear that h_* is an isomorphism, so we suppose K has dimension greater than zero. We choose a simplex σ of maximal dimension, and we let K_1 be the subcomplex of K all faces of σ let K_2 be the subcomplex of all simplices in K with the exception of σ. If $K_1 = K$, then $H_n(K) = 0$ for $n > 0$ and $H_0(K) = 0$, and similarly for the singular homology of $|K|$. It is easy to check in this case that h_* must be an isomorphism. Suppose instead that K does not consist of σ and its faces. Then $K = K_1 \cup K_2$, where K_1, K_2, and $K_1 \cap K_2$ all have fewer simplices than K, and we have a Mayer–Vietoris sequence for K, K_1, K_2. If we can show that we also have a Mayer–Vietoris sequence

$$\to H_n(|K_1| \cap |K_2|) \to H_n(|K_1|) \oplus H_n(|K_2|) \to H_n(|K|) \to H_{n-1}(|K_1| \cap |K_2|) \to \dots,$$

then by use of induction and the five-lemma, we may conclude that $H_n : H_n(K) \to H_n(|K|)$ is an isomorphism.

Unfortunately, $|K| = |K_1| \cup |K_2|$ is not a covering with the property that the interiors of the subspaces cover $|K|$. Hence, we must use a 'fattening' argument to get the desired sequence. Let $U_2 = |K_2| \cup (\sigma - \{b\})$ where b is the barycenter of σ. Since $|K_1| \cap |K_2| = \dot{\sigma}$ is a deformation retract of $\sigma - \{b\}$, it follows that $|K_2|$ is a deformation retract of U_2. Also, $|K|$ is the union of of the open set U_2 and the interior of $|K_1| =\overset{\circ}{\to} \sigma$. Finally, $|K_1| \cap U_2 = \sigma - \{b\}$.

It follows that we have a Mayer–Vietoris sequence

$$\to H_n(\sigma - \{b\}) \to H_n(|K_1|) \oplus H_n(U_2) \to H_n(|K|) \to H_{n-1}(\sigma - \{b\}) \to .$$

Now consider the diagram

$$\begin{tikzcd}
H_n(|K_1|\cap|K_2|) \arrow[r] \arrow[d] & H_n(|K_1|)\oplus H_n(|K_2|) \arrow[r] \arrow[d] & H_n(|K|) \arrow[r, "?"] \arrow[d] & H_{n-1}(|K_1|\cap|K_2|) \arrow[d] \\
H_n(\sigma-\{b\}) \arrow[r] & H_n(|K_1|)\oplus H_n(U_2) \arrow[r] & H_n(|K|) \arrow[r] & H_{n-1}(\sigma-\{b\})
\end{tikzcd}$$

where the morphism '?' has not been defined. Each vertical homomorphism is an isomorphism induced by inclusion. It follows that we may use the diagram to define the homomorphism '?' and the resulting sequence is exact.

To complete the proof, we need to consider the diagram

$$\begin{tikzcd}
H_n(K_1\cap K_2) \arrow[r] \arrow[d] & H_n(K_1)\oplus H_n(K_2) \arrow[r] \arrow[d] & H_n(K) \arrow[r] \arrow[d] & H_{n-1}(K_1\cap K_2) \arrow[d] \\
H_n(|K_1|\cap|K_2|) \arrow[r] & H_n(|K_1|)\oplus H_n(|K_2|) \arrow[r] & H_n(|K|) \arrow[r] & H_{n-1}(|K_1|\cap|K_2|)
\end{tikzcd}$$

As discussed above, we may assume inductively that all the vertical homomorphisms—except the desired one—are isomorphisms, so the five lemma will give the desired result, provided we know the diagram commutes. Consider the following diagram of chain complexes.

$$\begin{tikzcd}
0 \arrow[r] & C_*(K_1\cap K_2) \arrow[r] \arrow[d] & C_*(K_1)\oplus C_*(K_2) \arrow[r] \arrow[d] & C_*(K) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & S_*(|K_1|\cap U_2) \arrow[r] & S_*(|K_1|)\oplus S_*(U_2) \arrow[r] & S^{\mathcal{U}}(|K|) \arrow[r] & {}
\end{tikzcd}$$

where each sequence may be used to derive the appropriate Mayer–Vietoris seqeunce. Note that under $h_\sharp$, it is in fact true that $C_*(K)$ maps to $S^{\mathcal{U}}(|K|)$. This shows that the diagram with 'fattened' objects on the bottom row commutes. We leave it to the student to verify that replacing the 'fattened' row by the original row still yields a commutative diagram.

We have now proved

THEOREM 7.6. *The homomoprhism $h_* : H_*(K) \to H_*(|K|)$ is an isomorphism for every finite simplicial complex K.*

There is one point which was left unresolved. Namely, the homomorphism h_* appeared to depend on the order chosen for K. In fact it does not so depend. This may be proved by considering the homology of the chain complex $O_*(K)$. Namely, there is an obvious chain map $h'_\sharp : O_*(K) \to S_*(|K|)$; just map an ordered affine simplex (even with repetitions) to itself viewed as a singular simplex in $|K|$. We clearly have a factorization

Hence, to show that h_* does not depend on the ordering, it would suffice to show that the chain morphism $O_*(K) \to C_*(K)$ induces an isomorphism in homology. We won't do this in this course. If you want to see a proof, look in one of the texts on algebraic topology such as Munkres, Rotman, or Spanier. In any event, an important consequence of this result is the following

Proposition 7.7. *The homomorphism h_* is natural.*

The above arguments would have worked just as well for reduced homology. For relative homology, we may use the Five Lemma as follows. Let L be a subcomplex of K. $h_\sharp$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_*(L) \arrow[r] \arrow[d] & C_*(K) \arrow[r] \arrow[d] & C_*(K, L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(|L|) \arrow[r] & S_*(|K|) \arrow[r] & S_*(|K|, |L|) \arrow[r] & 0
\end{tikzcd}$$

which in turn yields the commutative diagram

$$\begin{tikzcd}
H_n(L) \arrow[r] \arrow[d] & H_n(K) \arrow[r] \arrow[d] & H_n(K, L) \arrow[r] \arrow[d] & H_{n-1}(L) \arrow[r] \arrow[d] & H_{n-1}(K) \arrow[d] \\
H_n(|L|) \arrow[r] & H_n(|K|) \arrow[r] & H_n(|K|, |L|) \arrow[r] & H_{n-1}(|L|) \arrow[r] & H_{n-1}(|K|)
\end{tikzcd}$$

Now use the Five Lemma to conclude

Corollary 7.8. *Let K be a finite simplicial complex and L a subcomplex. Then we have a natural isomorphism $H_*(K, L) \to H_*(|K|, |L|)$.*

5. Some Algebra. The Tensor Product

Let A and B be abelian groups. We shall define a group $A \otimes B$, called the *tensor product*. The tensor product plays an important role in what is called multilinear algebra. It is useful in subjects ranging from algebraic topology to differential geometry.

Here is the definition. Let $F(A, B)$ be the free abelian group with basis the set $A \times B$, i.e., the set of pairs (a, b) with $a \in A, b \in B$. In $F(A, B)$ consider the subgroup $T(A, B)$ generated by all elements of the form

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b) \qquad a_1, a_2 \in A, b \in B \tag{9}$$

$$(a, b_1 + b_2) - (a, b_1 + b_2) \qquad a \in A, b_1, b_2 \in B \tag{10}$$

Then, define $A \otimes B = F(A, B)/T(A, B)$.

Denote by $a \otimes b$ the element of $A \otimes B$ represented by $(a.b) \in A \times B$. Note that the elements $a \otimes b$ *generate* $A \otimes B$, but not every element is of that form. Generally, an element will be expressible $\sum_{i=1}^{k} a_i \otimes b_i$, but not necessarily in a unique way.

What we have done is to force the relations

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b \tag{11}$$

$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 \tag{12}$$

in $A \otimes B$. In fact, $A \otimes B$ is the largest possible group in which such relations hold. To explain this, first define $p : A \times B \to A \otimes B$ by $p(a, b) = a \otimes b$. Then the above formulas may be rewritten

$$p(a_1 + a_2, b) = p(a_1, b) + p(a_2, b) \tag{13}$$

$$p(a, b_1 + b_2) = p(a, b_1) + p(a, b_2). \tag{14}$$

A function $g : A \times B \to C$ is called *biadditive* if it satisfies these conditions, so we may say instead that p is biadditive.

Proposition 7.9. *Let A and B be abelian groups. If $g : A \times B \to C$ is a biadditive function, then there exists a unique group homomorphism $G : A \otimes B \to C$ such that the diagram*

$$\begin{tikzcd} A \times B \arrow[rr, "p"] \arrow[dr, "g"'] & & A \otimes B \arrow[dl, "G"] \\ & C & \end{tikzcd}$$

commutes.

Proof. There exists such a homomorphism. For, g induces a homomorphis $g_1 : F(A, B) \to C$, namely, $g_1(a, b) = g(a, b)$ specifies g_1 on basis elements. Since

$$g(a_1 + a_2, b) - g(a_1, b) - g(a_2, b) = 0 \tag{15}$$

$$g(a, b_1 + b_2) - g(a, b_1) - g(a, b_2) = 0 \tag{16}$$

it follows that $g_1(T(A, B)) = 0$. Thus, g_1 induces $G : A \otimes B = F(A, B)/T(A, B) \to C$ which clearly has the right properties.

Any such G is unique. For,

$$G(a \otimes b) = G(p(a, b)) = g(a, b)$$

determines G on a generating set for $A \otimes B$. □

The tensor product is a bit difficult to get at directly. The explicit definition starts with a very large free group, and the set of relations—i.e., the subgroup $T(A, B)$—is not particularly easy to compute with. Hence, one must usually determine tensor products from their properties. The most fundamental property is the universal mapping property asserted in the above Proposition.

5.1. Example. We claim that $\mathbf{Z}/n\mathbf{Z} \otimes \mathbf{Z}/m\mathbf{Z} = 0$ if $(n, m) = 1$. To see this, note that

$$n(a \otimes b) = a \otimes b + \cdots + a \otimes b \quad n \quad \text{times} \tag{17}$$

$$= (a + \cdots + a) \otimes b = (na) \otimes b = 0 \otimes b = 0. \tag{18}$$

(Can you prove $0 \otimes b = 0$?) Similarly, $m(a \otimes b) = 0$. Hence, both n and m kill every generator of $A \otimes B$, so they both kill $A \otimes B$. Since $1 = nr + ms$, it follows that every element of $A \otimes B$ is zero.

Proposition 7.10. *For any group A, we have $A \otimes \mathbf{Z} \cong A$.*

Proof. Define $A \times \mathbf{Z} \to A$ by $(a, n) \mapsto na$. It is clear that this is biadditive. Hence, it induces $j : A \otimes \mathbf{Z} \to \mathbf{Z}$ such that $j(a \otimes n) = j(p(a, b)) = na$. Define a homomorphism $i : A \to A \otimes \mathbf{Z}$ by $i(a) = a \otimes 1$. (It is not hard to check this is a homomorphism.) Some simple calculation shows that i and j are inverses. □

Note that the argument used in the above proof is often abbreviated as follows. 'Define $j : A \otimes \mathbf{Z} \to A$ by $j(a \otimes n) = na$'. For this to make sense, there has to be a 'overlying' biadditive map, but in writing '$j(a \otimes n) = na$', it is presumed that the reader has checked the biadditivity of what is on the right, so the formula makes sense. Of course, we can't in general specify $j(a \otimes b)$ in an arbitrary way and get a well defined homomorphism. The elements $a \otimes b$ form a set of generators for $A \otimes B$, but there are many relations among these generators which may not be preserved by what we want to set $j(a \otimes b)$ to.

PROPOSITION 7.11. *If A and B are abelian groups, then $A \otimes B \cong B \otimes A$.*

PROOF. Use $a \otimes b \mapsto b \otimes a$. Why does this work? □

The tensor product is our first example of a *bifunctor*. Namely, suppose $f : A \to A'$ and $g : B \to B'$ are homomorphisms of abelian groups. Then, it is easy to check that the map $A \times B \to A' \otimes B$ defined by $(a, b) \to f(a) \otimes g(b)$ is biadditive, so it induces a homomorphism $A \otimes B \to A' \otimes B'$ which is denoted $f \otimes g$. This homomorphism takes $a \otimes b = p(a, b)$ to the same $f(a) \otimes g(b)$, so $f \otimes g$ is characterized by the property

$$(f \otimes g)(a \otimes b) = f(a) \otimes g(b) \qquad a \in A, b \in B.$$

We leave it to the student to check that this is all functorial, i,e.,

$$(f_1 \circ f_2) \otimes (g_1 \circ g_2) = (f_1 \otimes g_1) \circ (f_2 \otimes g_2)$$

whenever it makes sense.

The tensor product is also an example of an additive functor, i.e., it is consistent with direct sums.

PROPOSITION 7.12. *Let $A_i, i \in I$ be a collection of abelian groups indexed by some set I and let B be an abelian group. Then*

$$(\bigoplus_i A_i) \otimes B \cong \bigoplus_i (A_i \otimes B).$$

PROOF. We have inverse homomorphisms

$$(a_i)_{i \in I} \otimes b \mapsto (a_i \otimes b)_{i \in I} \tag{19}$$

$$(a_i \otimes b)_{i \in I} \mapsto (a_i)_{i \in I} \otimes b \tag{20}$$

(Why are these well defined? You should think it out carefully. You might find it easier to understand if you consider the case where there are only two summands: $A = A' \oplus A''$, and each element of the direct sum is a pair (a', a'').) □

Tensor products *in certain cases* preserve exact sequences. First, we always have the following partial result.

THEOREM 7.13. *Suppose*

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$$

is an exact sequence of abelian groups and B is an abelian group. Then

$$A' \otimes B \xrightarrow{i\otimes \mathrm{Id}} A \otimes B \xrightarrow{j\otimes \mathrm{Id}} A'' \otimes B \longrightarrow 0$$

is exact.

This property is called *right exactness*.

PROOF. (i) $j \otimes \mathrm{Id}$ is an epimorphism. For, given a generator $a'' \otimes b$, it is the image of $a \otimes b$, where $j(a) = a''$.

(ii) $\mathrm{Ker}(j \otimes \mathrm{Id}) \supseteq \mathrm{Im}(i \otimes \mathrm{Id})$. For, $(j \otimes \mathrm{Id}) \circ (i \otimes \mathrm{Id}) = j \circ i \otimes \mathrm{Id} = 0 \otimes \mathrm{Id} = 0$.

(iii) $\mathrm{Ker}(j \otimes \mathrm{Id}) \subseteq \mathrm{Im}(i \otimes \mathrm{Id}) = I$. To see this, first define a homomorphism $j_1 : A'' \otimes B \to (A \otimes B)/I$ as follows. Define a function $A'' \times B \to (A \otimes B)/I$ by

$$(a'', b) \mapsto \overline{a \otimes b} \in (A \otimes B)/I$$

where $j(a) = a''$. This is a well defined map, since if $j(a_1) = j(a_2) = a''$, we have $j(a_1 - a_2) = 0$, so by the exactness of the original sequence, $a_1 - a_2 = i(a')$. Hence,

$$a_1 \otimes b - a_2 \otimes b = (a_1 - a_2) \otimes b = i(a') \otimes b \in \mathrm{Im}\, i \otimes \mathrm{Id} = I.$$

Thus,

$$\overline{a_1 \otimes b} = \overline{a_2 \otimes b} \in (A \otimes B)/I.$$

It is easy to check that this function is biadditive, so it induces the desired homomorphism j_1 satisfying

$$j_1(a'' \otimes b) = \overline{a \otimes b} \qquad \text{where } j(a) = a''.$$

This may be rewritten

$$j_1(j \otimes \mathrm{Id})(a \otimes b) = \overline{a \otimes b},$$

so we get the following commutative diagram

$$\begin{tikzcd}
A \otimes B \arrow[rr, "j\otimes \mathrm{Id}"] \arrow[dr] & & A'' \otimes B \arrow[dl, "j_1"] \\
& (A \otimes B)/I &
\end{tikzcd}$$

Thus,

$$\mathrm{Ker}(j \otimes \mathrm{Id}) \subseteq \mathrm{Ker}(j_1 \circ (j \otimes \mathrm{Id})) = I$$

as claimed. $\square$

5.2. Example. The sequence in the Theorem is not always exact at the left hand end. To see this, consider

$$0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

and tensor this with $\mathbf{Z}/2\mathbf{Z}$. We get

$$\mathbf{Z}/2\mathbf{Z} \xrightarrow{0} \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$$

so the homomorphism on the left is not a monomorphism.

However, the sequence always is exact on the left if the group B is torsion free.

THEOREM 7.14. *Suppose $i : A' \to A$ is a monomorphism of abelian groups and B is a torsion free abelian group. Then $i \otimes \mathrm{Id} : A' \otimes B \to A \otimes B$ is a monomorphism.*

PROOF. First, assume that B is free and finitely generated, i.e., $B = \bigoplus_j \mathbf{Z}$. Then

$$A' \otimes B \cong \bigoplus_j A' \otimes \mathbf{Z} \cong \bigoplus_j A'$$

and similarly

$$A \otimes B \cong \bigoplus_j A.$$

These isomorphisms are natural in an appropriate manner, so the conclusion follows from the fact that $\bigoplus_j A' \to \bigoplus_j A$ is a monomorphism.

Next, assume only that B is torsion free. Suppose $\sum_j a'_j \otimes b_j \in A' \otimes B$ is in the kernel of $i \otimes \mathrm{Id}$, i.e.,

$$\sum_j i(a'_j) \otimes b_j = 0 \in A \otimes B.$$

Note first that there are only a finite number of b_j in this sum. So in $A \otimes B = F(A, B)/T(A, B)$ only a finite number of basis elements $(i(a'_j), b_j)$ of $F(A, B)$ are needed to represent the left hand side of the above equation. Moreover, the assertion that the element is zero in $F(A, B)/T(A, B)$ means that

$$\sum_j ((a'_j), b_j) \in T(A, B)$$

which means that it is a linear combination of *finitely many* of the generators of the subgroup $T(A, B)$. Hence, only a finite number of elements of B are needed to represent the desired element *and* the fact that it is zero. Let B' be the subgroup of B generated by these elements. $\sum_j i(a'_j) \otimes b_j = 0 \in A \otimes B'$ by the choice of B'. However,

B' is finitely generated and torsion free. Hence, by the first part of the argument, $i \otimes \mathrm{Id} : A' \otimes B' \to A \otimes B'$ is a monomorphism. Hence, $\sum_j a'_j \otimes b_j = 0 \in A' \otimes B'$. Hence, $\sum_j a'_j \otimes b_j = 0 \in A' \otimes B$ as claimed. $\square$

The notion of tensor product is considerably more general than what we did here. If A and B are modules over a commutative ring R, then one may define the tensor product $A \otimes_R B$ in a manner similar to what we did above. The result is again an R-module. (There is an even more general definition for modules over non-commutative rings.) In this context, a module is called *flat* if tensoring with it preserves monomorphisms. We leave most of this discussion for your algebra course.

5.3. Applications to Rank. The tensor product may be used to reduce questions about finitely generated generated abelian groups to questions about vector spaces. We do this by tensoring with the abelian group $\mathbf{Q}$, the additive group of rational numbers. If A is any abelian group, the abelian group $A \otimes \mathbf{Q}$ may be given the structure of a vector space over $\mathbf{Q}$. Namely, for $c \in \mathbf{Q}$ define

$$c(a \otimes b) = a \otimes (cb) \qquad a \in A, b \in \mathbf{Q}.$$

There is more than meets the eye in this definition. The formula only tells us how to multiply generators $a \otimes b$ of $A \otimes \mathbf{Q}$ by rational numbers. To see that this extends to arbitrary elements of $A \otimes \mathbf{Q}$, it is necessary to make an argument as before about a bi-additive function. We shall omit that argument. It is also necessary to check that the distributive law and all the other axioms for a vector space over $\mathbf{Q}$ hold. We shall also omit those verifications. The student is encouraged to investigate these issues on his/her own.

Let $A = F \oplus T$ where F is free of rank r and T is a torsion group. Then

$$A \otimes \mathbf{Q} \cong F \otimes \mathbf{Q} \oplus T \otimes \mathbf{Q}.$$

Proposition 7.15. *If A is a torsion group, then $A \otimes \mathbf{Q} = 0$.*

Proof. Exercise. $\square$

It follows that $A \otimes \mathbf{Q} \cong F \otimes \mathbf{Q}$. Suppose $F = \bigoplus_{i=1}^r \mathbf{Z}x_i$ is free with basis $\{x_1, x_2, \dots, x_r\}$. Then, by additivity, we see that

$$A \otimes \mathbf{Q} \cong \bigoplus_{i=1}^{r} \mathbf{Z}x_i \otimes \mathbf{Q} \cong \bigoplus_{i} \mathbf{Q}(x_i \otimes 1).$$

Thus, the rank of A as an abelian group is just the dimension of $A \otimes \mathbf{Q}$ as a vector space over $\mathbf{Q}$.

6. The Lefschetz Fixed Point Theorem

As mentioned earlier, one advantage of using simplicial homology is that we can reduce calculations to a chain complex of finitely generated abelian groups. One example of this is the definition of the so-called *Euler characteristic*. Suppose X is a polyhedron. Then the Euler characteristic of X is

$$\chi(X) = \sum_i (-1)^i rank(H_i(X)).$$

Since $H_i(X) = 0$ outside a finite range, this sum makes sense. For example,

$$\chi(S^n) = 1 + (-1)^n \qquad n > 0 \tag{21}$$

$$\chi(T^2) = 1 - 2 + 1 = 0. \tag{22}$$

The integers $rank H_i(X)$ are called the *Betti numbers* of X, so $\chi(X)$ is the alternating sum of the Betti numbers.

The Euler characteristic derives some of its importance from the fact that it may be caculated directly from a simplicial complex triangulating X.

PROPOSITION 7.16. *Let K be a finite simplicial complex.*

$$\sum_i (-1)^i rank(C_i(K) = \sum_i (-1)^i H_i(K).$$

Note $H_i(|K|) = H_i(K)$. Also, $rank(C_i(K))$ is just the number of simplices in K of dimension i. Thus, triangulating T^2 as we did previously, we see that

$$\chi(T^2) = 9 - 27 + 18 = 0,$$

and we may make this calcultion without computing the homology groups of T^2.

PROOF. We have short exact sequences

$$0 \to Z_i(K) \to C_i(K) \to B_{i-1}(K) \to 0 \tag{23}$$

$$0 \to B_i(K) \to Z_i(K) \to H_i(K) \to 0. \tag{24}$$

Hence,

$$rank(C_i) = rank(Z_i) + rank(B_{i-1}) \tag{25}$$

$$rank(Z_i) = rank(B_i) + rank(H_i) \tag{26}$$

Thus,

$$\sum_i (-1)^i rank(C_i) = \sum_i (-1)^i B_i + \sum_i (-1)^i rank(H_i) + \sum_i (-1)^i rank(B_{i-1}) \tag{27}$$

$$= \sum_i (-1)^i rank(H_i). \tag{28}$$

□

We shall see later that it is not even necessary to decompose the space into simplices. Cells or more elaborate 'polyhedra' will do. In particular, suppose S^2 is decomposed into 'polygons' $F_1, F_2, \ldots, F_r$. In side each polygon, we may choose a point p_i and joining each p_i to each of the vertices of F_i will yield a triangulation of S^2. Suppose F_i has v_i vertices. Then the effect of doing this will be to replace one 'face' F_i by v_i triangles, to add v_i edges, and and the add one vertex. The net change in the Euler characteristic from the i-face is

$$1 - v_i + (v_i - 1) = 0.$$

We illustrate the use of the Euler characteristic by determining all regular solids in $\mathbf{R}^3$—the Platonic solids. Each of these may be viewed as a 3-ball with its boundary S^2 decomposed into f r-gons such that, at each vertex, k faces meet. Let e be the number of edges and v the number of vertices. Then since each face has r edges and each edge belongs to precisely two faces, we have $fr = 2e$. Also, since each vertex belongs to k edges, and each edge has 2 vertices, we have $kv = 2e$. Since the Euler characteristic of S^2 is 2, we have

$$v - e + f = 2 \tag{29}$$

$$2e/k - e + 2e/r = (1/k - 1/2 + 1/r)2e = 2 \tag{30}$$

$$(\frac{1}{k} + \frac{1}{r} - \frac{1}{2}) = 1 \tag{31}$$

Since $k, r \geq 3$, and since $1/k + 1/r - 1/2$ must be positive, the only possibilities are summarized in the following table

k	r	1/k + 1/r - 1/2	e	v	f	Solid
3	3	1/6	6	4	4	Tetrahedron
3	4	1/12	12	8	6	Cube
4	3	1/12	12	6	8	Octahedron
3	5	1/30	30	20	12	Dodecahedron
5	3	1/30	30	12	20	Icosohedron

There is an important generalization which applies to a self map $f : X \to X$ where X is a polyhedron. To discuss this we need some preliminary concepts. If $g : A \to A$ is an endomorphism of a finitely generated abelian group, we may consider the induced $\mathbf{Q}$-linear transformation $f \otimes \mathrm{Id} : A \otimes \mathbf{Q} \to A \otimes \mathbf{Q}$. We denote the *trace* of this linear transformation by $tr(g)$. Note that $tr(f) \in \mathbf{Z}$. For, if T is the torsion subgroup of A, g induces $\overline{g} : A/T \to A/T$ where A/T is free of finite rank. Since $T \otimes \mathbf{Q} = 0$, the epimorphism $A \otimes \mathbf{Q} \to (A/T) \otimes \mathbf{Q}$ is an isomorphism, so we may identify $g \otimes \mathrm{Id}$ with $\overline{g} \otimes \mathrm{Id}$. However, if we choose a basis for A/T over $\mathbf{Z}$, it will also be a basis for $(A/T) \otimes \mathbf{Q}$ over $\mathbf{Q}$. Hence, the matrix of $\overline{g} \otimes \mathrm{Id}$ will be the same as the (integer) matrix of $\overline{g}$. Hence, the trace will be an integer.

The trace is additive on short exact sequences, i.e.,

Lemma 7.17. *Let A be a finitely generated abelian group, A' a subgroup. Suppose g is an endomorphism of A such that $g(A') \subseteq A'$. Let g' be the restriction of g to A' and g'' the induced homomorphism on $A'' = A/A'$. Then $tr(g) = tr(g') + tr(g'')$.*

Proof. Consider the short exact sequence

$$0 \to A' \otimes \mathbf{Q} \to A \otimes \mathbf{Q} \to A'' \otimes \mathbf{Q} \to 0$$

of vector spaces. It suffices to prove that

$$tr(g \otimes \mathrm{Id}) = tr(g' \otimes \mathrm{Id}) + tr(g'' \otimes \mathrm{Id}).$$

Modulo some identifications, we may consider $A' \otimes \mathbf{Q}$ a $\mathbf{Q}$-subspace of $A \otimes \mathbf{Q}$ and $A'' \otimes \mathbf{Q}$ the resulting quotient space. By standard vector space theory, we can choose a basis for $A \otimes \mathbf{Q}$ such that the matrix of $g \otimes \mathrm{Id}$ with respect to this basis has the form

$$\begin{bmatrix} C' & 0 \\ * & C'' \end{bmatrix}$$

where C' is a matrix representation of $g' \otimes \mathrm{Id}$ and C'' is a matrix representation of $g'' \otimes \mathrm{Id}$. Taking traces yields the desired formula. $\square$

Let $f : X \to X$. Define the Lefschetz number of f to be

$$L(f) = \sum_i (-1)^i tr(H_i(f)).$$

By the above remarks, $L(f)$ is an integer.

6.1. Examples. Let $f : S^n \to S^n$ be a self map. Then

$$L(f) = 1 + (-1)^n deg(f).$$

This uses the fairly obvious fact that if X is path connected, and f is a self map, then $H_0(f)$ is the identity isomorphism of $H_0(X) = \mathbf{Z}$, so $tr(H_0(f) = 1$.

For any polyhedron, the Euler characteristic is the Lefschetz number of the identity map since $tr(H_i(\mathrm{Id})) = rank(H_i(X))$.

Like the Euler characteristic, the Lefschetz number may be calculated using simplicial chains. For, suppose $g : C_* \to C_*$ is a chain morphism of a chain complex inducing homomorphisms $H_i(g) : H_i(C_*) \to H_i(C_i)$ in homology. As before we have the exact sequences

$$0 \to Z_i \to C_i \to B_{i-1} \to 0 \tag{32}$$

$$0 \to B_i \to Z_i \to H_i \to 0 \tag{33}$$

and g will induce endomorphism $Z_i(g), C_i(g)$, and $B_i(g)$ of each of these groups. Then

$$tr(C_i(g)) = tr(Z_i(g)) + tr(B_{i-1}(g)) \tag{34}$$

$$tr(Z_i(g)) = tr(B_i(g)) + tr(H_i(g)). \tag{35}$$

Taking the alternating sum yields as in the case of the Euler characteristic

$$\sum_i (-1)^i tr(g_i) = \sum_i (-1)^i tr(H_i(g)) = L(g) \tag{36}$$

THEOREM 7.18. *(Lefschetz Fixed Point Theorem) Let $f : X \to X$ be a self map of a polyhedron. If $L(f) \neq 0$ then f has a fixed point.*

The following corollary is a generalization of the Brouwer Fixed Point Theorem

COROLLARY 7.19. *Let $f : X \to X$ be a self map of an acyclic polyhedron (i.e., $H_i(X) = 0, i > 0$). Then f has a fixed point*

PROOF. $L(f) = 1 \neq 0$. □

PROOF. Suppose f does not have a fixed point. We shall show that there is a triangulation K of X and a chain morphism $F_\sharp : C_*(K) \to C_*(K)$ which induces $H_*(f) : H_*(X) \to H_*(X)$ in homology, and such that for each simplex $\sigma \in K$, the chain $F_\sharp(\sigma)$ does not involve σ. That says that for each i, the matrix of the homomorphism $F_i : C_i(K) \to C_i(K)$ has zero diagonal entries. Hence, $tr(F_i) = 0$ so by formula (36), $L(f) = 0$.

The idea behind this argument is fairly clear. Since X is compact, if f does not have any fixed points, it must be possible to subdivide X finely enough into simplices so that no simplex is carried by f into itself. However, there are many technical complications in making this argument precise, and we shall now deal with them.

By picking a homeomorphic space, we may assume $X = |L|$ for some finite simplical complex (contained in an appropriate $\mathbf{R}^N$. Under the assumption that f has no fixed points, $|f(x) - x|$ has a lower bound. By taking sufficiently many subdivisions, so the mesh of L is small enough, we can assume

$$\overline{St_L(v)} \cap f(\overline{St_L(v)} = \emptyset \tag{37}$$

for every vertex in L.

Our first problem is that f won't generally come from a simplicial morphism of L, so we choose an iterated barycentric subdivision L' of L and a simplicial approximation $\phi : L' \to L$ to f. Thus,

$$f(\overline{St_{L'}(v')}) \subseteq \overline{St_L(\phi(v'))} \tag{38}$$

for every vertex v' in L'. Note that $|\phi|$ is homotopic to f, but it is not quite the simplicial map we want since it takes $C_*(L')$ to $C_*(L)$ instead of to $C_*(L')$. We shall deal with that problem below, but note for the moment that ϕ meets our needs in the following partial sense. Let σ be a simplex of L and suppose σ' is a simplex of L' which is contained in σ. Let v be any vertex of σ. Then

$$\sigma' \subseteq \sigma \subseteq \overline{St_L(v)},$$

it follows from (37) that $f(\sigma')$ is disjoint from $\overline{St_L(v)}$. Hence, by (38), we can't have $v = \phi(v')$ for any vertex v' of σ'. In other words, if σ' is a simplex of L' contained in a simplex σ of L, then

$$\phi(\sigma') \neq \sigma.$$

We now deal with the problem that $\phi_\sharp$ does not end up in $C_*(L')$. For this, we define a chain morphism $j_\sharp : C_*(L) \to C_*(L')$ which will have appropriate properties. This is done by barycentric subdivision. Namely, for each simplex σ in L define a chain $Sd(\sigma)$ in $C_*(SdL)$ exactly as we did for singular homology. This presumes that we have chosen a fixed order for the vertices of L and a consistent order for the vertices of SdL, but otherwise the definition (by induction) uses the same formulas as in the singular case. $j_\sharp$ will be consistent with the chain morphism to singular theory, i.e.,

$$\begin{tikzcd}
C_*(L) \arrow[r, "Sd_\sharp"] \arrow[d, "h_\sharp"'] & C_*(SdL) \arrow[d, "h_\sharp"] \\
S_*(X) \arrow[r, "Sd_\sharp"] & S_*(X)
\end{tikzcd}$$

commutes. Hence, in homology, we get a commutative diagram

$$\begin{tikzcd} H_*(L) \arrow[r, "Sd_*"] \arrow[d, "h_*"'] & H_*(SdL) \arrow[d, "h_*"] \\ H_*(X) \arrow[r, "="] & H_*(X) \end{tikzcd}$$

Here we use the fact that for singular theory, the subdivision operator is chain homotopic to the identity. If we iterate this process, we get the desired chain map $j_\sharp : C_*(L) \to C_*(L')$ and

$$\begin{tikzcd} H_*(L) \arrow[r, "j_*"] \arrow[d, "h_*"'] & H_*(L') \arrow[d, "h_*"] \\ H) * (X) \arrow[r, "="] & H_*(X) \end{tikzcd}$$

commutes.

To complete the proof, consider the diagram

$$\begin{tikzcd} H_*(L') \arrow[r, "\phi_*"] \arrow[d, "h_*"'] & H_*(L) \arrow[d, "h_*"] \\ H_*(X) \arrow[r, "|\phi|_*"] & H_*(X) \end{tikzcd}$$

This commutes by the naturality of the h_* homomorphism. (We did not prove naturality in general, but in the present case we can assume that the orderings needed to define the $h_\sharp$ morphisms are chosen so that ϕ is order preserving, in which case the commutativity of the diagarm is clear.) Putting this diagram alongside the previous one and using formula (36) shows that the chain morphism $j_\sharp \circ \phi_\sharp : C_*(L') \to C_*(L')$ may be used to calculate $L(|\phi|)$. However, by our assumption $\phi(\sigma')$ does not contain σ', so under barycentric subdivision, σ' does not appear in $j_\sharp(\phi(\sigma'))$, and as above each trace is zero. Since $f \sim |\phi|$, it follows that $L(f) = L(|\phi|) = 0$ as claimed.

$\square$

CHAPTER 8

Cell Complexes

1. Introduction

We have seen examples of ‘triangulations’ of surfaces for which the rule that two simplexes intersect in at most an edge fails. For example, the decomposition indicated in the diagram below of a torus into two triangles fails on that ground. However, if we ignore that fact and calculate the homology of the associated chain complex (using the obvious boundary), we will still get the right answer for the homology of a 2-torus. Even better, there is no particular reason to restrict our attention to triangles. We could consider the torus as a single ‘2-cell’ ϕ as indicated below with its boundary consisting of two ‘1-cells’ σ, τ, which meet in one ‘0-cell’ ν. Moreover the diagram suggests the following formulas for the ‘boundary’:

$$\partial_2\phi = \sigma + \tau - \sigma - \tau = 0$$
$$\partial_1\sigma = \partial_1\tau = \nu - \nu = 0.$$

From this it is easy to compute the homology: $H_0 = \mathbf{Z}\nu, H_1 = \mathbf{Z}\sigma \oplus \mathbf{Z}\tau$, and $H_2 = \mathbf{Z}\phi$. A decomposition of this kind (not yet defined precisely) is called a cellular decomposition, and the resulting structure is called a cell complex. The diagram below indicates how this looks with the torus imbedded in $\mathbf{R}^3$ in the usual way.

Indicated below is a calculation of the homology of $\mathbf{R}P^2$ by similar reasoning. Note that it is clear why there is an element of order two in H_1.

Here is a similar calculation for S^2 where we have one 2-cell ϕ, no 1-cells, and one 0-cell ν.

This same reasoning could be applied to S^n so as to visualize it as a cell complex with one n-cell and one 0-cell.

1.1. $\mathbf{C}P^n$. The definition given previously for $\mathbf{R}P^n$, real projective n-space, may be mimicked for any field F. Namely, consider the vector space F^{n+1}, and define an equivalence relation in $F^{n+1} - \{0\}$

$$x \sim y \Leftrightarrow \exists c \in F^* \quad \text{such that } y = cx,$$

and let FP^n be the quotient space of this relation. Thus FP^n consists of the set of *lines* or 1 dimensional linear subspaces in F^{n+1} suitably topologized. If $x = (x_0, \dots, x_n)$, the components $x_0, \dots, x_n$ are called *homogeneous coordinates* of the corresponding point of FP^n. As above, different sets of homogeneous coordinates for the same point differ by a constant, non-zero multiplier. Note also that FP^0 consists of a single point.

Let $F = \mathbf{C}$. Then the *complex projective space* $\mathbf{C}P^n$ is the quotient space of $\mathbf{C}^{n+1} - 0 \simeq \mathbf{R}^{2n+2} - 0$. We may also view it as a quotient of S^{2n+1} as follows. First note that by multiplying by an appropriate positive real number, we may assume that the homogeneous coordinates $(x_0, x_1, \dots, x_n)$ of a point in $\mathbf{C}P^n$ satisfy

$$\sum_{i=0}^{n} |x_i|^2 = 1,$$

so the corresponding point in $C^{n+1} - 0$ lies in S^{2n+1}. Furthermore, two such points, $(x'_0, \dots, x'_n)$ and $(x_0, \dots, x_n)$, will represent the same point in $\mathbf{C}P^n$ if and only if $x'_i = cx_i, i = 0, \dots, n$ with $|c| = 1$, i.e., $c \in S^1$. Hence, we can identify $\mathbf{C}P^n$ as the orbit space of the action of the group S^1 on S^{2n+1} defined through complex coordinates by

$$c(x_0, \dots, x_n) = (cx_0, \dots, cx_n).$$

The student should verify that the map $p : S^{2n+1} \to \mathbf{C}P^n$ is indeed a quotient map. It follows that $\mathbf{C}P^n$ is compact. We also need to know that it's Hausdorff.

PROPOSITION 8.1. *$\mathbf{C}P^n$ is compact Hausdorff.*

Proof. It suffices to prove that $p : S^{2n+1} \to \mathbf{C}P^n$ is a closed map. Then by Proposition 3.9 we know that $\mathbf{C}P^n$ is compact Hausdorff.

To show that p is closed consider a closed set $A \subset S^{2n+1}$ and look at the diagram

$$\begin{tikzcd} S^1 \times S^{2n+1} \arrow[r] & S^{2n+1} \\ S^1 \times A \arrow[u] \arrow[ur] & \end{tikzcd}$$

The horizontal map is the action of S^1 on S^{2n+1}, which is continuous. Since $S^1 \times A$ is compact, the image of $S^1 \times A$ under this action is compact, hence closed. But this image is just $p^{-1}(p(A))$, which is what we needed to show was closed. $\square$

We shall describe a cellular decomposition of $\mathbf{C}P^n$. First note that $\mathbf{C}P^{k-1}$ may be imbedded in $\mathbf{C}P^k$ through the map defined using homogeneous coordinates by

$$(x_0, x_1, \dots, x_{k-1}) \mapsto (x_0, x_1, \dots, x_{k-1}, 0).$$

With these imbeddings, we obtain a tower

$$\mathbf{C}P^n \supset \mathbf{C}P^{n-1} \supset \dots \supset \mathbf{C}P^1 \supset \mathbf{C}P^0.$$

We shall show that each of the subspaces $\mathbf{C}P^k - \mathbf{C}P^{k-1}, k = 1, \dots, n$ is homeomorphic to an open $2k$-ball in $\mathbf{R}^{2k}$. To prove this, assume as above that the homogeneous coordinates $(x_0, x_1, \dots, x_k)$ of a point in $\mathbf{C}P^k$ are chosen so $\sum_{i=0}^{k} |x_i|^2 = 1$. Let $x_k = r_k e^{i\theta_k}$ where $0 \leq r_k = |x_k| \leq 1$. By dividing through by $e^{i\theta_k}$ (which has absolute value 1). we may arrange for $x_k = r_k$ to be real and non=negative without changing the fact that $\sum_i |x_i|^2 = 1$. Then,

$$0 \leq x_k = \sqrt{1 - \sum_{i=0}^{k-1} |x_i|^2} \leq 1.$$

Let D^{2k} denote the closed $2k$-ball in $\mathbf{R}^{2k}$ defined by

$$\sum_{i=1}^{2k} y_i^2 \leq 1.$$

Define $f_k : D^{2k} \to \mathbf{C}P^k$ by

$$f_k(y_1, \dots, y_{2k}) = (x_0, \dots, x_k)$$

where $x_0 = y_1 + iy_2$, $x_1 = y_3 + iy_4$, $\dots$, $x_{k-1} = y_{2k-1} + iy_{2k}$ and $x_k = \sqrt{1 - |y|^2}$. The coordinates on the left are ordinary cartesian coordinates, and the coordinates on the right are homogeneous coordinates. f_k is clearly continuous and, by the above discussion f_k is onto. Since D^{2k} is compact and $\mathbf{C}P^k$ is Hausdorff, it follows that f_k is a closed map. We shall show that it is one-to-one on the open ball $D^{2k} - S^{2k-1}$, and it follows easily from this that it a homeomorphism on the open ball.

Suppose $f_k(y') = f_k(y)$, i.e.,

$$(x'_0, \ldots, x'_k) = c(x_0, \ldots, x_k) \qquad c \in S^1.$$

Suppose $|y'| < 1$. Then, $x'_k = \sqrt{1 - |y'|^2} > 0$, so it follows that c is real and positive, hence $c = 1$, which means $|y| < 1$ and $y' = y$. Note that this argument shows a little more. Namely, if two points of D^{2k} map to the same point of $\mathbf{C}P^k$, then they must both be on the boundary S^{2k-1} of D^{2k}.

We now investigate the map f_k on the boundary S^{2k-1}. First note that $|y| = 1$ holds if and only if $x_k = 0$, i.e., if and only $f_k(y) \in \mathbf{C}P^{k-1}$. Thus, f_k does map $D^{2k} - S^{2k-1}$ homeomorphically onto $\mathbf{C}P^k - \mathbf{C}P^{k-1}$ as claimed. Moreover, it is easy to see that the restriction of f_k to S^{2k-1} is just the quotient map described above taking S^{2k-1} onto $\mathbf{C}P^{k-1}$. (Ignore the last coordinate x_k which is zero.)

The above discussion shows us how to view $\mathbf{C}P^n$ as a cell complex (but note that we haven't yet defined that concept precisely.) First take a point σ_0 to be viewed as $\mathbf{C}P^0$. Adjoin to this a disk D^2 by identifying its boundary to that point. This yields $\mathbf{C}P^2$ which we see is homemorphic to S^2. Call this 'cell' σ_2. Now attach D^4 to this by mapping its boundary to σ_2 as indicated above; call the result σ_4. In this way we get a sequence of cells

$$\sigma_0 \subset \sigma_2 \subset \cdots \subset \sigma_{2n}$$

each of which is the quotient of a closed ball of the appropriate dimension modulo the equivalence relation described above on the boundary of the ball. In what follows we shall develop these ideas somewhat further, and show that the homology of such a cell complex may be computed by taking as chain group the free abelian group generated by the cells, and defining an appropriate boundary homomorphism. The definition of that boundary homomorphism is a bit tricky, but in the present case, since there are no cells in odd dimensions, the boundary homomorphism should turn out to be zero. That is, we should end up with the following chain complex for $\mathbf{C}P^n$:

$$\begin{aligned} C_{2i} &= \mathbf{Z} && 0 \le i \le n, \\ C_{2i-1} &= 0 && 1 \le i \le n, \quad \text{and} \\ \partial_k &= 0 && \text{all } k. \end{aligned}$$

Hence, *after we have finished justifying the above claims* we shall have proved the following assertion:

THEOREM 8.2. *The singular homology groups of $\mathbf{C}P^n$ are given by*

$$H_k(\mathbf{C}P^n) = \begin{cases} \mathbf{Z} & \text{if } k = 2i, 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

2. Adjunction Spaces

We now look into the idea of 'adjoining' one space to another through a map. We will use this to build up cell complexes by adjoining one cell at a time.

Let X, Y be spaces and suppose $f : A \to Y$ is a map with domain A a *subspace* of X. In the *disjoint union* $X \sqcup Y$ consider the equivalence relation generated by the basic relations $a \sim f(a)$ for $a \in A$. The quotient space $X \sqcup Y/\sim$ is called the adjunction space obtained by attaching X to Y through f. It is denoted $X \sqcup_f Y$. f is called the *attaching map.*

EXAMPLE 8.3. Let $f : A \to \{P\}$ be a map to a point. Then $X \sqcup_f \{P\} \simeq X/A$.

EXAMPLE 8.4. Let $X = D^2$ be the closed disk in $\mathbf{R}^2$ and let Y also be the closed disk. Let $A = S^1$ and let $f : A \to Y$ be the degree 2 map of S^1 onto the boundary of Y which is also S^1. Then $X \sqcup_f Y$ is a 2-sphere with antipodal points on its equator identified. Can you further describe this space?

Note that there is a slight technical problem in the definition. Since X and Y could in principle have points in common, the disjoint union $X \sqcup Y$ must be defined by taking spaces homeomorphic to X and Y but which are disjoint and then forming their union. That means that assertions like $a \sim f(a)$ don't technically make sense in $X \sqcup Y$. However, if one is careful, one may identify X and Y with the corresponding subspaces of $X \sqcup Y$. You should examine the above examples with these remarks in mind.

Let $\rho_Y : Y \to X \sqcup_f Y$ be the composite map

$$Y \xrightarrow{\iota} X \sqcup Y \xrightarrow{\rho} X \sqcup_f Y$$

where ι is inclusion into the disjoint union, and ρ is projection onto the quotient space. Define ρ_X analagously.

PROPOSITION 8.5. *(i) ρ_Y maps Y homeomorpically onto a subspace of $X \sqcup_f Y$.*

(ii) If A is closed then $\rho_Y(Y)$ is closed in $X \sqcup_f Y$.

(iii) If A is closed, then ρ_X is a homeomorphism of $X - A$ onto $X \sqcup_f Y - \rho_Y(Y)$.

Because of (i), we may identify Y with its image in $X \sqcup_f Y$. This is a slight abuse of terminology.

The map $\rho_X : X \to X \sqcup_f Y$ is called the *characteristic map* of the adjunction space. Because relations of the form $a \sim f(a)$ will imply realtions of the form $a_1 \sim a_2$ where $f(a_1) = f(a_2)$, it won't generally be true that X can be identified with a subspace of $X \sqcup_f Y$. Nevertheless, (iii) says that $X - A$ can be so identified. In general, a map of pairs $h : (X, A) \to (X', A')$ is called a *relative homeomorphism* if its restriction to $X - A$ is a homeomorphism onto $X' - A'$. Thus, if A is closed, ρ_X provides a relative homeomorphism of (X, A) with $(X \sqcup_f Y, Y)$.

PROOF. (i) Note first that two distinct elements of Y are never equivalent, so ρ_Y is certainly one-to-one. Let U be an open set in Y. Then $f^{-1}(U)$ is open in A, so $f^{-1}(U) = A \cap V$ for some open set V in X. $V \sqcup U$ is open in $X \sqcup Y$, and it is not hard to see that it is a union of equivalence classes of the relation $\sim$. That implies that its image $\rho(V \sqcup U)$ is open in $X \sqcup_f Y$. However, $\rho_Y(U) = \rho_Y(Y) \cap \rho(V \sqcup U)$, so $\rho_Y(U)$ is open in $\rho_Y(Y)$.

(ii) Exercise.

(iii) ρ_X is certainly one-to-one on $X - A$. Since $X - A$ is open in X, any open set U in $X - A$ is open in X. However, any subset of $X - A$ is a union of (singleton) equivalence classes, so it follows that

$$\rho(U \sqcup \emptyset) = \rho_X(U)$$

is open in in $X \sqcup_f Y$. However, this is contained in the open set $X \sqcup_f Y - Y$ of $X \sqcup_f Y$, so it is open in that. The fact that $\rho_X(X - A) = X \sqcup_f Y - Y$ is obvious. $\square$

We now turn our attention to a special case of fundamental importance, namely when $X = D^n$ and $A = S^{n-1}$. Given a map $S^{n-1} \to Y$, the resulting adjunction space $D^n \sqcup_f Y$ is referred to as the space obtained by attaching, or adjoining, an n-cell to Y.

The following result allows us to recognize when a subspace of a space is the result of adjoining an n-cell.

PROPOSITION 8.6. *Let Y be a compact Hausdorff space, e, S disjoint subspaces with S closed. Suppose there is a map*

$$\phi : (D^n, S^{n-1}) \to (e \cup S, S)$$

which is a relative homeomorphism. Then $D^n \sqcup_f S \simeq e \cup S$ where $f = \phi|S^{n-1}$.

PROOF. Since $D^n \sqcup S$ is compact and Y is Hausdorff, it suffices by Proposition 3.6 to show that for the map $D^n \sqcup S \to e \cup S$, the preimages of points are the equivalence classes of the relation in $D^n \sqcup S$ generated by $x \sim f(x), x \in S^{n-1}$. This is clear since ϕ is one-to-one on $D^n - S^{n-1}$. $\square$

EXAMPLE 8.7. Let $X = D^{2n}, A = S^{2n-1}$, and let $f : S^{2n-1} \to \mathbf{C}P^{n-1}$ be the map described in the previous section. Then $D^{2n} \sqcup_f S^{2n-1} \simeq \mathbf{C}P^n$. For the discussion in the previous section establishes that the hypothesis of Proposition 8.6 is satisfied.

We now want to study the effect on homology of adjoining an n-cell D^n to a space Y through an attaching map $f : S^{n-1} \to Y$. Take U to be the open set in $D^n \sqcup_f Y - Y$ which is homeomorphic to the open cell $D^n - S^{n-1}$ through the characteristic map ϕ. Take $V = D^n \sqcup_f Y - \{\phi(0)\}$. Then $D^n \sqcup_f Y = U \cup V$, and $U \cap V$ is homeomorphic to an open ball less its center, so it has S^{n-1} as a deformation retract.

LEMMA 8.8. *Y is a deformation retract of V.*

PROOF. Define a retraction $r : V \to Y$ by

$$r(z) = \begin{cases} z & \text{if } z \in Y, \\ \phi(z/|z|) & \text{if } z \in D^n - \{0\}. \end{cases}$$

To show that r is deformation retraction, we define a homotopy $F : V \times I \to V$ as follows:

$$F(z,t) = \begin{cases} z & \text{if } z \in Y, \\ \phi((1-t)z + tz/|z|) & \text{if } z \in D^n - \{0\}. \end{cases}$$

Usually, we just assert that it is obvious that a map is continuous, but in this case, because of what happens on the boundary of D^n, it is not so obvious. Consider the following diagram

$$\begin{tikzcd}
((D^n - \{0\}) \sqcup Y) \times I \arrow[r, "\tilde{F}"] \arrow[d, "\rho\times\mathrm{Id}"'] & (D^n - \{0\}) \sqcup Y \arrow[d, "\rho"] \\
V \times I \arrow[r, "F"] & V
\end{tikzcd}$$

where $\tilde{F}$ is defined in the obvious way on each component of the disjoint sum. It is easy to check that the diagram commutes and that the vertical map on the right (induced from the quotient map $D^n \sqcup Y \to D^n \sqcup_f Y$) is a quotient map. If we knew that the vertical map on the left were a quotient map, then it would follow that F is continuous (check this!). Unfortunately the product of a quotient map with the the identity map of a space needn't be a quotient map, but it's true in this case because I is a nice space. We quote the following result about quotient spaces and products. A proof can be found, for example, in Munkres, *Elements of Algebraic Topology*, Theorem 20.1.

PROPOSITION 8.9. *Let $\rho : X \to Y$ be a quotient map. Suppose Z is any locally compact Hausdorff space. Then $\rho \times \mathrm{Id} : X \times Z \to Y \times Z$ is a quotient map.*

By the above remarks, this completes the proof of Lemma 8.8. $\square$

Suppose $n > 0$. The conditions for a Mayer–Vietoris sequence apply to $U \cup V$, so we have a long exact sequence

$$\cdots \to \tilde{H}_i(S^{n-1}) \to \tilde{H}_i(Y) \to \tilde{H}_i(D^n \sqcup_f Y) \to \tilde{H}_{i-1}(S^{n-1}) \to \ldots$$

For $i \neq n, n-1$, this yields

$$\tilde{H}_i(Y) \cong \tilde{H}_i(D^n \sqcup_f Y).$$

For $i \neq 0$, the $\sim$'s are not needed, and in fact, it is not hard to see that they are not needed for $i = 0$. Thus, we have essentially proved the following

THEOREM 8.10. *Let $n > 0$ and let $f : S^{n-1} \to Y$ be a map. Then, for $i \neq n, n-1$,*

$$H_i(Y) \cong H_i(D^n \sqcup_f Y).$$

For $i = n, n-1$, we have an exact sequence

$$0 \to H_n(Y) \to H_n(D^n \sqcup_f Y) \to \mathbf{Z} \to H_{n-1}(Y) \to H_{n-1}(D^n \sqcup_f Y) \to 0.$$

We can now determine the homology groups of complex projective spaces without explicit use of 'cellular chains', but of course that idea is implicit in the argument. We repeat the statement in the previous section.

THEOREM 8.11. *The singular homology groups of* $\mathbf{C}P^n$ *are given by*

$$H_k(\mathbf{C}P^n) = \begin{cases} \mathbf{Z} & \text{if } k = 2i, 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Use $\mathbf{C}P^n = D^{2n} \sqcup_f \mathbf{C}P^{n-1}$. The corollary is true for $n = 0$. By the above discussion $H_i(\mathbf{C}P^n) \cong H_i(\mathbf{C}P^{n-1})$ for $i \ne 2n, 2n-1$. For $i = 2n, 2n-1$, we have

$$0 \to H_{2n}(\mathbf{C}P^{n-1}) \to H_{2n}(\mathbf{C}P^n) \to \mathbf{Z} \to H_{2n-1}(\mathbf{C}P^{n-1} \to H_{2n-1}(\mathbf{C}P^n) \to 0.$$

Since $H_{2n}(\mathbf{C}P^{n-1}) = H_{2n-1}(\mathbf{C}P^{n-1}) = 0$, it follows that $H_{2n-1}(\mathbf{C}P^n) = 0$ and $H_{2n}(\mathbf{C}P^n) = \mathbf{Z}$. □

EXAMPLE 8.12 (Products of Spheres). For $m, n > 0$,

$$S^m \times S^n \simeq D^{m+n} \sqcup_f (S^m \vee S^n)$$

for an appropriate attaching map f.

Model D^m by I^m. Then D^{m+n} is modelled by $I^{m+n} \simeq I^m \times I^n$. It is not hard to check the formula

$$\partial(I^m \times I^n) = \partial I^m \times I^n \cup I^m \cup \partial I^n$$

which is a set theoretic version of the product formula from calculus. Choose points $p_m \in S^m$ and $p_n \in S^n$, say the 'north poles' of each. Let $f_m : I^m \to S^m$ be a map which takes the interior of I^m onto $S^m - \{p_m\}$ and ∂I^m onto p_m. Then $f_m \times f_n$ maps I^{m+n} onto $S^m \times S^n$ and ∂I^{m+n} onto

$$(\{p_m\} \times S^n) \cup (S^m \times \{p_n\}) \simeq S^m \vee S^n.$$

(Note that $(\{p_m\} \times S^n) \cap (S^m \times \{p_n\}) = \{(p_m, p_n)\}$.) Thus, we have a mapping of pairs

$$f_m \times g_m : (I^{m+n}, \partial I^{m+n}) \to (S^m \times S^n, S^m \vee S^n)$$

as required by Proposition 8.6.

COROLLARY 8.13. *Let* $m, n > 0$. *If* $m < n$, *then*

$$H_i(S^m \times S^n) = \begin{cases} \mathbf{Z} & \text{if } i = 0, \\ \mathbf{Z} & \text{if } i = m, n, \\ \mathbf{Z} & \text{if } i = m + n, \\ 0 & \text{otherwise.} \end{cases}$$

If $m = n$, *then the only difference is that* $H_m(S^m \times S^m) = \mathbf{Z} \oplus \mathbf{Z}$.

Proof. Assume $m < n$. By a straightforward application of the Mayer–Vietoris sequence, we see that $H_i(S^m \vee S^n) = \mathbf{Z}$ if $i = 0, m, n$ and is zero otherwise.

Assume $m \neq 1$. Then, $H_i(S^m \times S^n) = \mathbf{Z}$ for $= 0, m, n$, it is zero otherwise except for the cases $i = m+n, m+n-1$. These cases are determined by considering

$$\begin{aligned} 0 \to H_{m+n}(S^m \vee S^n) = 0 \to H_{m+n}(S^m \times S^n) \\ \to \mathbf{Z} \to H_{m+n-1}(S^m \vee S^n) = 0 \to H_{m+n-1}(S^m \times S^n) \to 0. \end{aligned}$$

We see that $H_{m+n}(S^m \times S^n) = \mathbf{Z}$ and $H_{m+n-1} = 0$. The remaining cases $m = 1 < n$ and $m = n$ are left as excercises. □

Example 8.14 (Real Projective spaces). The diagram below illustrates the construction of $\mathbf{R}P^n$ by a scheme similar to that described previously for $\mathbf{C}P^n$.

We conclude

Proposition 8.15. *$\mathbf{R}P^n \simeq D^n \sqcup_f \mathbf{R}P^{n-1}$ for the usual (attaching) map $f : S^{n-1} \to \mathbf{R}P^{n-1}$.*

For example, $\mathbf{R}P^1 = D^2 \sqcup_f \mathbf{R}P^0 = S^1$.

Note that we cannot use the method used for $\mathbf{C}P^n$ and in the previous example to compute $H_*(\mathbf{R}P^n)$. The reason is that there are cells in every dimension. (You should try using the argument to see what goes wrong.)

Example 8.16 (Quaternionic Projective Spaces). It is known that there are precisely three real division algebras, $\mathbf{R}$, $\mathbf{C}$, and the quaternion algebra $\mathbf{H}$. The quaternion algebra is not commutative, but every non-zero element is invertible. We remind you of how it is defined. $\mathbf{H} = \mathbf{R} \times \mathbf{R}^3 = \mathbf{R}^4$ as a real vector space. Think of elements of $\mathbf{H}$ as pairs $(a, \mathbf{v})$ where $a \in \mathbf{R}$ and $\mathbf{v}$ is a 3-dimensional vector. The product in $\mathbf{H}$ is defined by

$$(a, \mathbf{u})(b, \mathbf{v}) = (ab - \mathbf{u} \cdot \mathbf{v}, a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v}).$$

Any element in $\mathbf{H}$ may be written uniquely $a + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -1 \\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k} \\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i} \\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}. \end{aligned}$$

These rules together with the distributive law determine the product.

Define $|(a, \mathbf{v})| = \sqrt{a^2 + |\mathbf{v}|^2}$ which is of course just the usual norm in $\mathbf{R}^4$. Then in addition to the usual rules, we have

$$|xy| = |x||y| \qquad x, y \in \mathbf{H}.$$

The set of non-zero quaternions forms a group under multiplication, and the set $S^3 = \{x \mid |x| = 1\}$ is a subgroup. (In fact, S^1 and S^3 are the only spheres which have group structures making them topological groups.)

Quaternionic projective space $\mathbf{H}P^n$ is defined to be the set of 1-dimensional $\mathbf{H}$-subspaces of $\mathbf{H}^{n+1}$. Let $(x_0, \dots, x_n) \in \mathbf{H}^{n+1} - \{0\} = \mathbf{R}^{4n+4} - \{0\}$ be homogenous quaternionic coordinates representing a point in $\mathbf{H}P^n$. By dividing by $\sqrt{\sum_i |x_i|^2}$ we may assume this point lies in S^{4n+3}. In fact, two points in S^{4n+3} represent the same point if and only if they differ by a quaternion mutiple x of norm 1, i.e., if and only if they are in the same orbit of the action of S^3 on S^{4n+3} given by $x(x_0, \dots, x_n) = (xx_0, \dots, xx_n)$. Then, reasoning as before, we have

$$\mathbf{H}P^n \simeq D^{4n} \sqcup_f \mathbf{H}P^{n-1}.$$

COROLLARY 8.17.

$$H_k(\mathbf{H}P^n) = \begin{cases} \mathbf{Z} & \text{if } k = 4i, 0 \le i \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Reason as in the case of $\mathbf{C}P^n$. □

Note. There is one other similar example. Namely, there is an 8 dimensional real algebra called the Cayley numbers which is a division algebra in the sense that every non-zero element is invertible. It is not an associative algebra. In any case, one can define projective n-'space' over the Cayley numbers and determine its homology groups. In approximately 1960, Adams showed that this is as far as one can go. There are no other real, possibly non-associative algebras in which every non-zero element is invertible. The proof uses arguments from algebraic topology although the result is purely algebraic.

3. CW Complexes

Let X be a space. A subspace e of X homeomorphic to $D^n - S^{n-1}$ is called an *open n-cell*. For $n = 0$, we interpret S^{-1} to be the empty set, so an 'open' zero cell is a point.

Let $\mathcal{C}$ be a collection of *disjoint* open cells e in X of various dimensions. Let X^k denote the union of all cells in $\mathcal{C}$ of dimension $\leq k$. X^k is called the k-skeleton of X. We have a tower

$$\emptyset = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^{k-1} \subset X^k \subseteq \ldots X$$

A *CW complex* is a pair $(X, \mathcal{C})$ where X is a Hausdorff space, $\mathcal{C}$ is a collection of open cells in X such that for each e in $\mathcal{C}$, there is a map

$$\phi_e : (D^k, S^{k-1}) \to (e \cup X^{k-1}, X^{k-1})$$

where k is the dimension of e, and the following rules hold.

(1) The cells in $\mathcal{C}$ are disjoint and X is their union.

(2) Each map ϕ_e is a relative homeomorphism.

(3) The closure $\overline{e}$ of each cell in $\mathcal{C}$ is contained in the union of finitely many cells in $\mathcal{C}$. (Closure finiteness.)

(4) A set in X is closed if and only if its interesection with $\overline{e}$ is closed for every cell e in $\mathcal{C}$. (The topology on X is called the weak topology relative to the collection of cells if this is true.) The name 'CW' is an abbreviation of the terminology for conditions (3) and (4). The concept was introduced by J. H. C. Whitehead.

If there is such a pair for X, we call X a CW complex without explicitly mentioning the family of cells and maps. Note that if $\mathcal{C}$ is finite, then conditions (3) and (4) are automatic. However, there are important CW complexes which are not finite. If $X = X^k$ for some k, the smallest such k is called the dimension of X. It is the largest dimension of any cell in $\mathcal{C}$.

The general idea is that a CW complex is a space that can be built up inductively by successively adjoining new cells in each dimension. For many purposes they form the most interesting class of topological spaces to study.

Example 8.18. Every simplicial complex K is a CW complex. Let $X = |K|$ and let $\mathcal{C}$ be the family of all $e = \sigma - \dot{\sigma}$ for σ a simplex in K. For each k-simplex, there is a homeomorphism $\phi_\sigma : D^k \to \sigma$ and that provides a relative homeomorphism

$$\phi_e : (D^k, S^{k-1}) \to (\sigma, \dot{\sigma}) \hookrightarrow (e \cup X^{k-1}, X^{k-1}).$$

Because of the conditions for a simplicial complex, the rules for a CW complex hold.

EXAMPLE 8.19. $\mathbf{R}P^n, \mathbf{C}P^n$, and $\mathbf{H}P^n$ are all finite CW complexes. Consider the family $\mathbf{R}P^n$. There is a natural injection of $\mathbf{R}P^n$ in $\mathbf{R}P^{n+1}$ such that the former space is the n-skeleton of the latter and there is one more open cell of dimension n. We have then an ascending chain of topological spaces

$$\mathbf{R}P^0 \subset \mathbf{R}P^1 \subset \mathbf{R}P^2 \subset \cdots \subset \mathbf{R}P^n \subset \ldots$$

and we may form the union, which is denoted $\mathbf{R}P^\infty$. This set has a cell structure with one cell in each dimension, and we may make it a topological space using the weak topology relative to this family of cells. The result is an infinite CW complex.

Similar constructions apply in the other two cases to form $\mathbf{C}P^\infty$ and $\mathbf{H}P^\infty$. For $\mathbf{C}P^\infty$ there is one cell in each even dimension and for $\mathbf{H}P^\infty$ one cell in each dimension divisible by 4.

EXAMPLE 8.20. There are two interesting CW structures to put on S^n.

The first is very simple. Let p_n denote the point $(0, \ldots, 1)$ (the north pole). Then $S^n - \{p_n\}$ is an open n-cell and $\{p_n\}$ is an open 0-cell. These are the only cells.

The second is related to our construction of $\mathbf{R}P^n$. There are two cells in each dimension. In dimension n the open upper hemisphere is one open cell and the lower open hemsiphere is the other. Their common boundary is a closed cell homeomorphic to S^{n-1}. Repeat this for S^{n-1} and continue iteratively down to $k = 0$.

EXAMPLE 8.21. As discussed in the previous section, $S^m \times S^n$ has a CW structure with one cell each in dimensions $m + n, m, n$, and 0. If $m = n$, there are two cells in dimension m.

A CW complex is called regular if each characteristic map provides a homeomorphism $D^k \to \overline{e}$, and $\overline{e} - e$ is a finite union of open cells of dimension less than k rather than just being contained in such a union.

4. The Homology of CW complexes

Let X be a CW complex and let e be an open k-cell in X. $\phi_e(D^k)$ is compact, so, since X is Hausdorff, it is closed. Since by (2), $\phi_e(D^k - S^{k-1}) = e$, it is not hard to check that $\phi_e(D^k) = \overline{e}$ and $\phi_e(S^{k-1}) = \overline{e} - e$. The latter is contained, by assumption, in X^{k-1} so by condition (3) in the definition, it is contained in a finite union of cells in $\mathcal{C}$ of dimension $< k$. Also, since $\overline{e}$ is compact, Proposition 8.6 on recognizing adjunction spaces shows that $\overline{e} = D^k \sqcup_{f_e} (\overline{e} - e)$.

PROPOSITION 8.22. *Let $(X, \mathcal{C})$ be a CW complex. Let $\mathcal{C}'$ be a subset of $\mathcal{C}$ with the property that for every cell e in $\mathcal{C}'$, $\overline{e}$ is contained in a finite union of cells in $\mathcal{C}'$. Then the union X' of the cells in $\mathcal{C}'$ is a closed subspace of X, and $(X', \mathcal{C}')$ is a CW complex.*

We call $(X', \mathcal{C}')$ a sub-CW-complex, or just a subcomplex. Note that it follows from this that the k-sketeton of a CW complex is closed and is a subcomplex.

PROOF. Exercise. □

PROPOSITION 8.23. $H_*(D^k, S^{k-1}) \cong H_*(\overline{e}, \overline{e} - e)$. *In particular,* $H_i(\overline{e}, \overline{e} - e) = \mathbf{Z}$ *for* $i = k$ *and it is zero otherwise.*

This lemma gives us a way to start counting cells through homology.

PROOF. The second statement follows from what we know about the relative homology of (D^k, S^{k-1}). (Use the long exact homology sequence.)

To demonstrate the isomorphism, use the argument we applied previously to study the homology of adjunction spaces. Let $E^k = D^k - S^{k-1}$, $s = \overline{e} - e$. Consider the diagram, in which we want to prove that the left hand vertical map is an isomorphism.

$$\begin{tikzcd}
H_i(D^k, S^{k-1}) \arrow[r, "1"] \arrow[d, "2"] & H_i(D^k, D^k - \{0\}) \arrow[d, "3"] & H_i(E^k, E^k - \{0\}) \arrow[l, "5"'] \arrow[d, "6"] \\
H_i(\overline{e}, s) \arrow[r, "4"] & H_i(\overline{e}, \overline{e} - \{p\}) & H_i(e, e - \{p\}) \arrow[l, "7"']
\end{tikzcd}$$

where the morphisms 2 and 3 come from $\phi = \phi_e$ and $p = \phi(0)$. 1 and 4 are isomorphisms. For 1, consider the long exact sequences of the pairs $(D^k, S^{k-1}$ and $(D^k - \{0\}, S^{k-1})$, and the morphisms between making appropriate diagrams commute. (Draw those diagrams if you are not sure of the argument.) $H_i(S^{k-1}) \to H_i(D^k - \{0\})$ is an isomorphism because the space on the left is a deformation retract of the space on the right. Of course the identity homomorphism $H_i(D^k) \to H_i(D^k)$ is an isomorphism. Now apply to five lemma to conclude that 1 is an isomorphism. A similar argument works for 4. (s is a deformation retract of $\overline{e} - \{p\}$ because $\overline{e}$ is an appropriate adjunction space.) It follows that we need only prove that 3 is an isomorphism.

The map 5 is an isomorphism because we may excise S^{k-1} and 7 is an isomorphism because we may excise s. Finally, 6 is an isomorphism because $E^k \to e$ is a homeomorphism. □

Now that we know how to isolate one cell homologically, we want to apply this wholesale to the collection of all open k cells.

For each k cell e in X, let $\phi_e : D^k_e \to X$ be a characteristic map for $\overline{e}$ where we label D^k to keep the domains separate. The collection of these maps gives us a map

$$\phi : \coprod_{e \in \mathcal{C}^k} D^k_e \to X$$

which carries $\sqcup_e S^{k-1}_e$ into X^{k-1}. Denote by f the restriction of ϕ to that subspace.

PROPOSITION 8.24. $(\coprod_{e\in\mathcal{C}^k} D^k_e) \sqcup_f X^{k-1} \simeq X^k$

PROOF. Let ρ denote the quotient map of $\coprod_e D^k_e \sqcup X^{k-1}$ onto $(\coprod_e D^k_e) \sqcup_f X^{k-1}$, and let Φ be the obvious map from the former disjoint union into X. By seeing what Φ does on equivalence classes, we conclude that there is a unique map $\tilde{\phi}$ from the adjunction space into X such that $\Phi = \tilde{\phi} \circ \rho$. Moreover, this map is certainly one-to-one and onto X^k. Hence, it suffices to show that it is a closed map. Take a saturated closed set in $(\bigcup_e D^k_e) \sqcup X^{k-1}$, i.e., one consisting of equivalence classes. Such a set is necessarily a disjoint union of closed sets of the form $\sqcup_e Y_e \sqcup Z$ where $f^{-1}(Z) = \sqcup_e (Y_e \cap S^{k-1}_e)$.

Because X^k is a CW complex, it suffices to prove that the image under Φ of this set intersects the closure of every cell in X^k in a closed set. Since no matter where we consider it, Z is closed in X^{k-1} which itself is closed, it follows that the intersections for cells of dimension less than k are all closed. Fix an open k cell e'. It remains to show that $\Phi(\sqcup_e Y_e)$ intersects $\overline{e}'$ in a closed set. However, $\Phi(Y_e) \cap \overline{e}'$ is already contained in a finite union of cells of dimension less that k for $e \neq e'$, so we need only consider the case $e = e'$. However, in that case we already know that $\phi_e(Y_k)$ is closed in $\overline{e}$ because ϕ_e is a quotient map. □

THEOREM 8.25. *Let X be a CW complex. $\tilde{\phi}$ induces and isomorphism*

$$\bigoplus_{e \in \mathcal{C}^k} H_i(D_e^k, S_e^{k-1}) \to H_i(X^k, X^{k-1}).$$

In particular, $H_k(X^k, X^{k-1})$ is free on a basis in one-to-one correspondence with the set of open k cells, and $H_i(X^k, X^{k-1}) = 0$ for $i \neq k$.

PROOF. The (relative) singular homology of a disjoint union is certainly the direct sum of the homologies of the factors. To prove the theorem, we just mimic the proof in the case of one k-cell. In particular consider the diagram

$$\begin{tikzcd}
(\sqcup_e D_e^k, \sqcup S_e^{k-1}) \arrow[r] \arrow[d] & (\sqcup D_e^k, \sqcup D_e^k - \{0_e\}) \arrow[d] & (\sqcup E_e^k, \sqcup E_e^k - \{0_e\}) \arrow[l] \arrow[d] \\
(X^k, X^{k-1}) \arrow[r] & (X^k, X^k - \cup_e\{p_e\}) & (\cup_e e, \cup(e - \{p_e\})) \arrow[l]
\end{tikzcd}$$

The only point worth mentioning about the argument is that X^{k-1} is a deformation retract of $X^k - \{p_e \mid e \in \mathcal{C}^k\}$. The argument depends as before on understanding the product of a quotient space with I.

There have been quite a few details omitted from this proof, which you might try to verify for yourself. In so doing, you will have to give names to some maps and untangle some identifications implicit in the above discussion. □

Let $(X, \mathcal{C})$ be a CW complex. Define $C_k(X) = H_k(X^k, X^{k-1})$. By the above result, it is free with basis the set of open cells of dimension k. We shall define a boundary morphism $\partial_k : C_k(X) \to C_{k-1}(X)$ such that the homology of the complex $C_*(X)$ is the singular homology of X. (This terminology seems to conflict with the notation for simplicial complexes, but since the latter theory may be subsumed under the theory of CW complexes, we don't have to worry about that.)

The boundary map is the connecting homomorphism in the homology sequence of the triple (X^k, X^{k-1}, X^{k-2})

$$H_k(X^{k-1}, X^{k-2}) \to H_k(X^k, X^{k-2}) \to H_k(X^k, X^{k-1}) \xrightarrow{\partial_k} H_{k-1}(X^{k-1}, X^{k-2}) \to \dots$$

PROPOSITION 8.26. $\partial_k \circ \partial_{k+1} = 0$.

PROOF. The following commutative diagram arises from the obvious map of triples $(X^k, X^{k-1}, X^{k-2}) \to (X^{k+1}, X^{k-1}, X^{k-2})$

$$\begin{tikzcd}
H_{k+1}(X^{k+1}, X^k) \arrow[r, "\partial_{k+1}"] & H_k(X^k, X^{k-1}) \arrow[r, "i_k"] \arrow[d, "\partial_k"] & H_k(X^{k+1}, X^{k-1}) \arrow[dl, "\partial'_k"] \\
& H_{k-1}(X^{k-1}, X^{k-2}) &
\end{tikzcd}$$

Since the composite homomorphism across the top is trivial, the result follows. ☐

THEOREM 8.27. *Let X be a CW complex. Then* $H_*(C_*(X)) \cong H_*(X)$

PROOF. The top row of the diagram (CD) above extends to the right with

$$H_k(X^k, X^{k-1}) \xrightarrow{i_k} H_k(X^{k+1}, X^{k-1}) \to H_k(X^{k+1}, X^k) = 0$$

so i_k is an epimorphism. Similarly, the vertical column on the right may be extended upward

$$\begin{tikzcd}
H_k(X^{k-1}, X^{k-2}) = 0 \arrow[d] \\
H_k(X^{k+1}, X^{k-2}) \arrow[d, "j_k"'] \\
H_k(X^{k+1}, X^{k-1})
\end{tikzcd}$$

so j_k is a monomorphism. A bit of diagram chasing should convince you that i_k maps $H_k(X^{k+1}, X^{k-2})$ (monomorphically) onto $j_k(\ker \partial_k) \cong \ker \partial_k / \operatorname{Im} d_{k+1} = H_k(C_*(X))$. Hence,

$$H_k(X^{k+1}, X^{k-2}) \cong H_k(C_*(X)).$$

It remains to prove that the left hand side of the above isomorphism is isomorphic to $H_k(X)$. First note that because $H_k(X^{k-i}, X^{k-i-1}) = H_{k-1}(X^{k-i}, X^{k-i-1}) = 0$ for $i \geq 2$, it follows that

$$H_k(X^{k+1}, X^{k-2}) \cong H_k(X^{k+1}, X^{k-3}) \cong \ldots \cong H_k(X^{k+1}, X^{-1}) = H_k(X^{k+1}).$$

Similarly, $H_{k+1}(X^{n+1}, X^n) = H_k(X^{n+1}, X^n) = 0$ for $n \geq k+2$ so $H_k(X^n) \to H_k(X^{n+1})$ is an isomorphism in that range. Hence,

$$H_k(X^{k+1}) \to H_k(X^n)$$

is an isomorphism for $n \geq k+1$. If X is a finite CW complex or even one of bounded dimension, then we are done.

In the general case, we need a further argument. Let u be a cycle representing an element of $H_k(X)$. Since only finitely many singular simplices occur in u, it comes from a cycle in some *compact* subspace of X. However, any compact subset of a CW complex lies in some n skeleton so we may assume u is actually a singular cycle representing an element in $H_k(X^n)$ for some n and clearly we loose nothing by assuming $n \geq k+1$. This together with the above argument shows $H_k(X^{k+1}) \to H_k(X)$ is onto. Suppose now that u is a cycle representing an element of $H_k(X^{k+1})$ which maps to zero in $H_k(X)$. Then $u = \partial_{k+1} v$ for some singular $k+1$-chain in X. As above, we may assume u, v are singular chains in some X^n for some $n \geq k+1$. That means u represents zero in $H_k(X^n)$. Since $H_k(X^{k+1}) \to H_k(X^n)$ is an isomorphism, it represents zero in $H_k(X^{k+1})$. That completes the proof.

Note the above isomorphisms are natural in the sense that a cellular map between CW complexes will yield appropriate commutative diagrams. This follows because everything is made up from natural homomorphisms in homology diagrams. The way the CW structure enters is in the characterization of the filtration of the space by k skeltons. A cellular map will carry one such filtration into the other. $\square$

We may use the above result to calculate $H_k(\mathbf{R}P^n)$ as follows. First consider the CW complex on S^n described above with two open cells e_k, f_k in each dimension $k = 0, 1 \dots, n$. With this decomposition, the k skeleton of S^n may be identified with S^k for $k = 0, \dots, n$. By the above theory, in that range,

$$C_k(S^n) = H_k(S^k, S^{k-1}) \cong \mathbf{Z}e_k \oplus \mathbf{Z}f_k.$$

There is one slight subtlety here. The copies of $\mathbf{Z}$ are obtained from the characteristic maps

$$(D^k, S^{k-1}) \to (\overline{e}_k, \overline{e}_k - e_k)$$
$$(D^k, S^{k-1}) \to (\overline{f}_k, \overline{f}_k - f_k)$$

so they are the images of

$$H_k(\overline{e}_k, \overline{e}_k - e_k) \to H_k(S^k, S^{k-1})$$
$$H_k(\overline{f}_k, \overline{f}_k - f_k) \to H_k(S^k, S^{k-1})$$

respectively. However, the generators of these summands are only uniquely determined modulo sign.

In any case, this decomposition is natural with respect to cellular maps. Consider in particular the antipode map $a_k : S^k \to S^k$. Clearly, this interchanges the two cells, and we may assume in the representation of $C_k(X)$ that $f_k = a_k(e_k), a_k(f_k) = e_k$.

Consider next the boundary homomorphism $\partial_k : C_k(S^n) \to C_{k-1}(S^n)$ This has to commute with the antipode map by naturality. Start in dimension 0. Choose a map $\pi : S^n \to \{P\}$ to a point. We may choose $e_0 \in H_0(S^0, S^{-1}) = H_0(S^0)$ so that $\pi_*(e_0)$ is a specific generator of $H_0(\{P\}) \cong \mathbf{Z}$, and clearly $\pi_*(f_0) = \pi_*(a_{0*}(e_0)$ is also that generator. Hence, in $H_0(S^n)$, we must have $e_0 \sim f_0$ so $e_0 - f_0$ must be a boundary in $C_0(S^n)$. Let $\partial_1 e_1 = xe_0 - yf_0$. Then $xe_0 \sim yf_0 \sim ye_0$ implies that $(x - y)e_0 \sim 0$. Since $H_0(S^n) = \mathbf{Z}$, it has no elements of finite order, so $x - y = 0$, i.e., $x = y$. Hence, $\partial_1 e_0 = x(e_0 - f_0)$ and applying the antipode map, we see $\partial_1 f_1 = x(f_0 - e_0)$. Hence, $\partial_1(ue_1 + vf_1) = x(u - v)(e_0 - f_0)$. Thus the only way that $e_0 - f_0$ could be a boundary is if $x = \pm 1$. By changing the signs of both e_1 and f_1 if necessary, we may assume $\partial_1 e_1 = e_0 - f_0, \partial_1 f_1 = f_0 - e_0$.

Consider next dimension 1. $e_1 + f_1$ is a cycle. However, $H_1(S^n) = 0$ (at least if $n > 1$.) It follows that $e_1 + f_1$ is a boundary. Let $\partial_2 e_2 = xe_1 + yf_1$. Subtracting off $y(e_1 + f_1)$ shows that $(x - y)e_1$ is a boundary, hence a cycle, but that is false unless $x = y$. Reasoning as above, we can see that $x = \pm 1$ and again we may assume it is 1. Thus, $\partial_2 e_2 = \partial_2 f_2 = e_1 + f_1$.

This argument may be iterated to determine all the ∂_k for $k = 1, 2, \ldots, n$. We conclude that

$$\begin{aligned} \partial_k e_k &= e_{k-1} + (-1)^k f_{k-1} \\ \partial_k f_k &= f_{k-1} + (-1)^k e_{k-1}. \end{aligned}$$

Now consider $\mathbf{R}P^n$ with the CW structure discussed previously, one open cell $\bar{e}_k$ in each dimension $k = 0, \ldots, n$. Using the cellular map $S^n \to \mathbf{R}P^n$, we may write $C_k(\mathbf{R}P^n) = \mathbf{Z}c_k$ where c_k is the image of e_k (and also of f_k). (See the Exercises.) Then by naturality, we find that for $k = 1, \ldots, n$,

$$\begin{aligned} \partial_k c_k &= (1 + (-1)^k)c_{k-1} \\ &= \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2c_{k-1} & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

We have now established the following

THEOREM 8.28.

$$H_k(\mathbf{R}P^n) = \begin{cases} \mathbf{Z} & \textit{if } k = 0, \\ \mathbf{Z}/2\mathbf{Z} & \textit{if } k \textit{ is odd, } 0 < k < n, \\ \mathbf{Z} & \textit{if } k = n \textit{ and } k \textit{ is odd,} \\ 0 & \textit{otherwise} \end{cases}$$

For example, $H_1(\mathbf{R}P^3) = \mathbf{Z}/2\mathbf{Z}, H_2(\mathbf{R}P^3) = 0, H_3(\mathbf{R}P^3) = \mathbf{Z}$, and $H_1(\mathbf{R}P^4) = \mathbf{Z}/2\mathbf{Z}, H_2(\mathbf{R}P^4) = 0, H_3(\mathbf{R}P^4) = \mathbf{Z}/2\mathbf{Z}, H_4(\mathbf{R}P^4) = 0$.

CHAPTER 9

Products and the Künneth Theorem

1. Introduction to the Künneth Theorem

Our aim is to understand the homology of the cartesian product $X \times Y$ of two spaces. The Künneth Theorem gives a complete answer relating $H_*(X \times Y)$ to $H_*(X)$ and $H_*(Y)$, but the answer is a bit complicated. One important special case says that *if* $H_*(X)$ *and* $H_*(Y)$ *are free*, then

$$H_*(X \times Y) = H_*(X) \otimes H_*(Y).$$

Example 9.1 (Products of Spheres). We saw in the previous chapter that $H_i(S^m \times S^n)$ is $\mathbf{Z}$ in dimensions $0, n, m, m+n$ if $m \neq n$ and $\mathbf{Z} \oplus \mathbf{Z}$ in dimension $n = m$ when the two are equal. This is accounted for by the Künneth Theorem as follows. Write

$$\begin{aligned} H_*(S^m) &= \mathbf{Z}e_0^m \oplus \mathbf{Z}e_m^m \\ H_*(S^n) &= \mathbf{Z}e_0^n \oplus \mathbf{Z}e_n^n \end{aligned}$$

where the subscript of each generator indicates its degree (dimension). If we take the tensor product of both sides and use the additivity of the tensor product and the fact that $\mathbf{Z} \otimes \mathbf{Z} = \mathbf{Z}$, we obtain

$$\begin{aligned} H_*(S^m) \otimes H_*(S^n) &= (\mathbf{Z}e_0^m \oplus \mathbf{Z}e_m^m) \otimes (\mathbf{Z}e_0^n \oplus \mathbf{Z}e_n^n) \\ &= \mathbf{Z}e_0^m \otimes e_0^n \oplus \mathbf{Z}e_0^m \otimes e_n^n \oplus \mathbf{Z}e_m^m \otimes e_0^n \oplus \mathbf{Z}e_m^m \otimes e_n^n. \end{aligned}$$

We see then how the answer is constructed. The rule is that if we tensor something of degree r with something of degree s, we should consider the result to have *total degree* $r + s$. With this convention, the terms in the above sum have degrees $0, n, m, m+n$ as required. Note also that we don't have to distinguish the $m = n$ case separately since in that case, there are two summands of the same total degree $n = 0 + n = n + 0$.

The above example illustrates the more explicit form of the Künneth Theorem in the free case

$$H_n(X \times Y) = \bigoplus_{r+s=n} H_r(X) \otimes H_s(Y).$$

If X and Y are CW complexes, it is not hard to see how such tensor products might arise. Namely, $C_*(X)$ and $C_*(Y)$ are each free abelian groups on the open cells of the respective CW complexes. Also, $X \times Y$ has a CW complex structure with open cells of the form $e \times f$ where e is an open cell of X and f is an open cell of Y. Hence,

$$C_*(X \times Y) = \bigoplus_{e,f} \mathbf{Z} e \times f \cong \bigoplus_{e,f} \mathbf{Z} e \otimes \mathbf{Z} f \cong \bigoplus_{e} \mathbf{Z} e \otimes \bigoplus_{f} \mathbf{Z} f \cong C_*(X) \otimes C_*(Y).$$

Note also that if $\dim(e \times f) = \dim e + \dim f$ which is consistent with the rule enunciated for degrees.

Unfortunately, the above decomposition is not the whole answer because we still have to relate $H_*(C_*(X) \otimes C_*(Y))$ to $H_*(C_*(X)) \otimes H_*(C_*(Y)) = H_*(X) \otimes H_*(Y)$. This comparison itself requires considerable work, i.e., we need a Künneth Theorem for chain complexes *before* we can derive such a theorem for spaces.

The analysis for CW complexes is not complete because we have not discussed the boundary homomorphism in $C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$. It turns out that this is fairly straightforward provided we know the boundary homomorphism for each term, but the latter morphisms are not easy to get at in general. (Of course, in specific cases, they are easy to compute, which is one thing that makes the CW theory attractive.) For theoretical purposes, we know that the singular chain complex is the 'right' thing to use because of its functorial nature. Unfortunately, it is not true in general that the 'product' of two simplices is again a simplex.

In fact, we spent considerable time studying how to simplicially decompose the product of an n-simplex and a 1-simplex, i.e, a prism, when proving the homotopy axiom. In general, in order to make use of singular chains, we need to relate $S_*(X \times Y)$ to $S_*(X) \otimes S_*(Y)$. It turns out that these are not isomorphic but there are chain maps between them with compositions which are chain homotopic to the identity. Thus, the two chain complexes are chain homotopy equivalent, and they have the same homology. This relationship is analyzed in the *Eilenberg–Zilber Theorem.*

Our program then is the following: first study the homology of the tensor product of chain complexes, then prove the Eilenberg–Zilber

Theorem, and then apply these results to obtain the Künneth Theorem for the product of two spaces.

2. Tensor Products of Chain Complexes

Let C' and C'' denote chain complexes. We make the tensor product $C' \otimes C''$ into a chain complex as follows. Define

$$(C' \otimes C'')_n = \bigoplus_{r+s=n} C'_r \otimes C''_s.$$

Also define boundary homomorphisms $\partial_n : (C' \otimes C'')_n \to (C' \otimes C'')_{n-1}$ by

$$\partial_n(x' \otimes x'') = \partial'_r x' \otimes x'' + (-1)^r x' \otimes \partial''_s x''$$

where $x' \in C'_r$ and $x'' \in C''_s$ and $r + s = n$. Note that the expression on the right is biadditive in x' and x'', so the formula does defined a homomorphism of $X'_r \otimes X''_s$. Since $(C' \otimes C'')_n$ is the direct sum of all such terms with $r + s = n$, this defines a homomorphism.

Proposition 9.2. $\partial_n \circ \partial_{n+1} = 0$.

Proof. Exercise. In doing this notice how the sign comes into play. □

When you do the calculation, you will see that the sign is absolutely essentially to prove that $\partial_n \circ d_{n+1} = 0$. We can also see on geometric grounds why such a sign is called for by considering the following example. Let $C' = C_*(I)$ where I is given a CW structure with two 0-cells and one 1-cell as indicated below. Similarly, let $C'' = C_*(I^2)$ with four 0-cells, four 1-cells and one 2-cell. We may view $C' \otimes C''$ as the chain complex of the product CW complex $I^3 = I \times I^2$. The diagram indicates how we expect the orientations and boundaries to behave. Note how the boudaries of various product cells behave. Note in particular how the boundary of $e'_1 \times e''_2$ ends up with a sign.

Our immediate problem then is to determine the homology of $C' \otimes C''$ in terms of the homology of the factors. First note that there is a natural homorphism

$$\times : H_*(C') \otimes H_*(C'') \to H_*(C' \otimes C'')$$

defined as follows. Given $z' \in \mathbf{Z}(C')$ of degree r represents $\overline{z'} \in H_r(C')$ and $z'' \in \mathbf{Z}(C'')$ of degree s represents $\overline{z''} \in H_s(C'')$ define

$$\overline{z'} \times \overline{z''} = \overline{z' \otimes z''} \in H_{r+s}(C' \otimes C'').$$

Note that the right hand side is well defined because

$$\begin{aligned}(z' + \partial' c') \otimes (z'' + \partial'' c'') &= z' \otimes z'' + \partial' c' \otimes z'' + z' \otimes \partial'' c'' + \partial' c' \otimes \partial'' c'' \\ &= z' \otimes z'' + \partial(c' \otimes z'' \pm z' \otimes c'' + c' \otimes \partial'' c'').\end{aligned}$$

It is also easy to check that it is biadditive, so

$$\overline{z'} \otimes \overline{z''} \to \overline{z'} \times \overline{z''}$$

defines a homomorphism $H_r(C') \otimes H_s(C'') \to H_{r+s}(C' \otimes C'')$ as required. We shall call this homomorphism the *cross product*, and as above we shall denote it by infix notation rather than the usual functional prefix notation.

We shall show that under reasonable circumstances, the homomophism is a monomorphism, and if $H_*(C')$ or $H_*(C'')$ is free then it is an isomorphism.

Suppose in all that follows that C' and C'' are free abelian groups, i.e., free $\mathbf{Z}$-modules. Then tensoring with any component C'_r or C''_s of either is an exact functor, i.e., it preserves short exact sequences. The importance of this fact is that any exact functor preserves homology. In particular, if A is a free abelian group and C is a chain complex, then $C \otimes A$ is also a chain complex (with boundary $\partial \otimes \mathrm{Id}$) and

$$H_*(C) \otimes A \cong H_*(C \otimes A).$$

The isomorphism is provided by $\overline{z} \otimes a \mapsto \overline{z \otimes a}$. It is clear that the isomorphism in natural with respect to morphisms of chain complexes and homomorphisms of groups.

Consider the exact sequence

$$0 \to Z(C'') \to C'' \to C''/Z(C'') \to 0. \tag{39}$$

This of course yields a collection of short exact sequences of groups, one in each degree s, but we may also view it as a short exact seqeunce of chain complexes, where $Z(C'')$ is viewed as a subcomplex of C'' with *zero* boundary homomorphism. Note also that since $\partial'' C'' \subseteq Z(C'')$, the quotient complex also has zero boundary. In fact, $C''/Z(C'')$ may be identified with $B(C'')$ (also with zero boundary) but with degrees shifted by one, i.e., $(C''/Z(C''))_r = B(C'')_{r-1}$. We shall denote the shifted complex by $B_+(C'')$.

Tensor the sequence 39 with C' to get the exact sequence of chain complexes

$$0 \to C' \otimes Z(C'') \to C' \otimes C'' \to C' \otimes B_+(C'') \to 0. \tag{40}$$

Note that each of these complexes is a tensor product complex, but for the two complexes on the ends, the contribution to the boundary from the second part of tensor product is trivial, e.g., $\partial_{p+q} c' \otimes z'' = \partial_p c' \otimes z''$. Since $Z(C'')$ and $B_+(C'')$ are also free, we have by the above remark

$$H_*(C') \otimes Z(C'') \cong H_*(C' \otimes Z(C''))$$
$$H_*(C') \otimes B_+(C'') \cong H_*(C' \otimes B_+(C'')).$$

Note however that because these are actually tensor products of complexes, we must still keep track of degrees, i.e., we really have

$$\bigoplus_{r+s=n} H_r(C') \otimes Z_s(C'') \cong H_n(C' \otimes Z(C''))$$
$$\bigoplus_{r+s=n} H_r(C') \otimes B_{+s}(C'') \cong H_n(C' \otimes B_+(C'')).$$

Now consider the long exact sequence in homology of the SES 40

$$\dots \xrightarrow{\delta_{n+1}} H_n(C' \otimes Z(C'')) \to H_n(C' \otimes C'')$$
$$\to H_n(C' \otimes B_+(C'')) \xrightarrow{\delta_n} H_{n-1}(C' \otimes Z(C'')) \to \dots$$

From this sequence, we get a SES

$$0 \to \operatorname{Coker}(\delta_{n+1}) \to H_n(C' \otimes C'') \to \operatorname{Ker}(\delta_n) \to 0. \tag{41}$$

We need to describe δ_{n+1} and δ_n in order to determine these groups. By the above discussion, $H_n(C' \otimes Z(C''))$ may be identified with the direct sum of components

$$H_r(C') \otimes Z_s(C'') \cong H_r(C' \otimes Z_s(C''))$$

where $r + s = n$. Similarly, $H_{n+1}(C' \otimes B_+(C''))$ may be identified with the direct sum of components of the form

$$H_r(C') \otimes B_{+,s+1}(C'') \cong H_r(C') \otimes B_s(C'')$$

where $r + s + 1 = n + 1$. Fix a pair, r, s with $r + s = n$.

Lemma 9.3. *δ_{n+1} on $H_r(C') \otimes B_s(C'')$ is just $(-1)^r \operatorname{Id} \otimes i_s$ where i_s is the inclusion of $B_s(C'')$ in $Z_s(C'')$.*

Proof. We just have to trace through the various identifications and definitions. Let $\overline{z'} \otimes \partial'' c''$ be a typical image we want to apply δ_{n+1} to. This should first be identified with $\overline{z' \otimes \partial'' c''}$. This is of course represented by $z' \otimes \partial'' c'' \in Zr + s + 1(C' \otimes B_+(C''))$. However, this comes from $z' \otimes c'' \in (C' \otimes C'')_{r+s+1}$, so taking its boundary, we obtain

$$\partial(z' \otimes c'') = (-1)^r z' \otimes \partial'' c''.$$

Not so surprisingly, this comes from a cycle in $(C' \otimes Z(C''))_{r+s}$, namely $(-1)^r z' \otimes \partial'' c''$. However, this is just what we want. □

Suppose now that $H_*(C'')$ is free. Then, for each s, the ses

$$0 \to B_s(C'') \to Z_s(C'') \to H_s(C'') \to 0$$

splits. Hence, since tensor products preserve direct sums, the sequence

$$0 \to H_r(C') \otimes B_s(C'') \xrightarrow{\delta_{n+1}} H_r(C') \otimes Z_s(C'') \to H_r(C') \otimes H_s(C'') \to 0$$

also splits; hence it is certainly exact. This yields two conclusions: $\operatorname{Coker}(\delta_{n+1}) \cong H_r(C') \otimes H_s(C'')$ and $\operatorname{Ker} \delta_n = 0$. Hence, it follows that

$$\bigoplus_{r+s=n} H_r(C') \otimes H_s(C'') \cong H_n(C' \otimes C'').$$

If you trace through the argument, you will find that the isomorphism is in fact given by the homomorphism '$\times$' defined earlier.

Clearly, we could have worked the argument with the roles of C' and C'' reversed if it were true that $H_*(C'')$ were free.

Theorem 9.4 (Kúnneth Theorem, restricted form). *Let C' and C'' be free chain complexes such that either $H_*(C')$ is free or $H_*(C'')$ is free. Then*

$$\times : H_*(C') \otimes H_*(C'') \to H_*(C' \otimes C'')$$

is an isomorphism.

In the next section, we shall determine $\operatorname{Coker} \delta_{n+1}$ and $\operatorname{Ker} \delta_n$ in the case neither $H_*(C')$ nor $H_*(C'')$ is free.

2.1. A Digression. There is another useful way to thing of the chain complex $C' \otimes C''$. It has two boundary homomorphisms $d' = \partial' \otimes \mathrm{Id}, d'' = \pm \mathrm{Id} \otimes \partial''$ where the sign $\pm = (-1)^r$ is given as above. These homomorphisms satisfy the rules

$$d'^2 = d''^2 = 0, \quad d'd'' + d''d' = 0.$$

What we have is the first important example of what is called a *double complex.*

We may temporarily set the first boundary in this chain complex to zero, so taking its homology amounts to taking the homology with respect the the second boundary. Since C' is free, tensoring with it is exact, and we have

$$H_*(C' \otimes C'', d'') \cong C' \otimes H_*(C'').$$

d' induces a boundary on this chain complex, and it makes sense to take the boundary on $H_*(C'')$ to be zero. Denote the reulst

$$H_*(C' \otimes H_*(C'')).$$

Note that it would be natural to conclude that this is $H_*(C') \otimes H_*(C'')$, but this is wrong except in the case $H_*(C')$ or $H_*(C'')$ is free as above. To study this quantity in general, consider as above the exact sequence of chain complexes (with trivial boundaries)

$$0 \to B(C'') \to Z(C'') \to H(C'') \to 0.$$

Tensoring as above with the free complex C' yields the exact sequence of chain complexes.

$$0 \to C' \otimes B(C'') \to C' \otimes Z(C') \to C' \otimes H(C'') \to 0.$$

This yields a long exact sequence in homology

$$\begin{aligned} H_n(C' \otimes B(C'')) \to H_n(C' \otimes Z(C')) \to H_n(C' \otimes H(C'')) \\ \to H_{n-1}(C' \otimes B(C'') \to H_{n-1}(C' \otimes Z(C'')) \to \dots \end{aligned}$$

However, as above

$$\begin{aligned} H_n(C' \otimes B(C'')) &\cong H_n(C') \otimes B(C'') \\ H_n(C' \otimes Z(C'')) &\cong H_n(C'') \otimes Z(C'') \end{aligned}$$

so we get an exact sequence

$$0 \to \operatorname{Coker}(\delta_{n+1}) \to H_n(C' \otimes C'') \to \operatorname{Ker}(\delta_n) \to 0. \tag{42}$$

You might conclude from this that this is the same sequence as 40 above, i.e., that the middle terms are isomorphic. In fact they are but only because, as we shall see, both sequences split. Thus

$$H_n(C' \otimes C'') \cong H_n(C' \otimes H(C''))$$

but *there is no natural isomorphism between them.*

3. Tor and the Künneth Theorem for Chain Complexes

To determine the homology of $C' \otimes C''$, we came down to having to determine the kernel and cokernel of the homorphism

$$\operatorname{Id} \otimes i_s : H_r(C') \otimes B_s(C'') \to H_r(C') \otimes Z_s(C'')$$

which (except for a sign) is the r, s component of δ_{r+s+1}. Thus, in effect we need to know what happens to the exact sequence

$$0 \to B_s(C'') \to Z_s(C'') \to H_s(C'') \to 0$$

when we tensor with $H_r(C')$.

Recall in general that if $0 \to B' \to B \to B'' \to 0$ is a ses of abelian groups, and A is any abelian group, then

$$A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$$

is exact. What we need to know for the Künneth Theorem is the cokernel and the kernel of the homomorphism on the left. It is clear from the right exactness that the cokernel is always isomorphic to $A \otimes B''$, but we don't yet have a way to identify the kernel. (We know the kernel is trivial if A is torsion free or if the 'B' sequence splits.) We shall define a new functor $\mathrm{Tor}(A, B)$ which will allow us to continue the seqeunce to the left:

(43)

$$\mathrm{Tor}(A, B') \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, B'') \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0.$$

It is pretty clear that this functor $\mathrm{Tor}(A, B)$ should have certain properties:

(i) It should be a functor of both variables.

(ii) If A is torsion free, if would make sense to require that $\mathrm{Tor}(A, -) = 0$, because that would insure that tensoring with A is an exact functor.

(iii) It should preserve direct sums.

(iv) Since $A \otimes B \cong B \otimes A$, we should have $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$. More generally, its properties should be symmetric in A and B.

(v) A sequence like 43 and its analogue with the roles of the arguments reversed should hold.

Property (ii) suggests the following approach. Let B be an abelian groups and choose a free presentation of it

$$0 \to R \xrightarrow{i} F \xrightarrow{j} B \to 0,$$

i.e., pick an epimorphism of a free abelians groups $j : F \to B$, and let R be the kernel. R is also free because *any subgroup of a free abelian group is free.* Define

$$\mathrm{Tor}(A, B) = \mathrm{Ker}(\mathrm{Id} \otimes i : A \otimes R \to A \otimes F).$$

Note that whatever the definition, if (ii) and (v) hold, we have an exact seqeunce

$$0 \to 0 \to \mathrm{Tor}(A, B) \to A \otimes R \to A \otimes F \to A \otimes B \to 0$$

so in any event $\mathrm{Tor}(A, B) \cong \mathrm{Ker}(\mathrm{Id} \otimes i)$. However, there is clearly a problem with this definition: *it appears to depend on the choice of presentation.*

THEOREM 9.5. *There is a functor* $\mathrm{Tor}(A, B)$ *such that for each presentation* $0 \to R \xrightarrow{i} F \to B \to 0$, *there is an isomorphism* $\mathrm{Tor}(A, B) \cong \mathrm{Ker}\,\mathrm{Id} \otimes i$. *Furthermore this isomorphism is natural with respect to homomorphisms of both arguments* A, B *and also with respect to morphisms of presentations.*

PROOF. It will be helpful to describe presentations from a slightly different point of view. Given a presentation, of B construct a chain complex with two non-zero groups by putting $Q_0 = F, Q_1 = R$ and $d_1 = i$. Then j may be viewed as a morphism of chain complexes $f \to B$ where the latter is the trivial chain complex with only one non-zero group, B, in degree 0. Then j induces an isomorphism of homology $H_*(Q) \cong B$, i.e., $H_0(Q) = B$, and $H_k(Q) = 0$ otherwise. Now consider the chain complex $A \otimes Q$. We have a natural isomorphism $H_0(A \otimes Q) \cong A \otimes B$, and $H_1(A \otimes Q)$ is supposed to be isomorphic to $\mathrm{Tor}(A, B)$.

LEMMA 9.6. *Let* Q *be a chain complex for a presentation of* B *and* Q' *a chain complex for a presentation of* B'. *Let* $f : B \to B'$ *be a homomorphism. Then there is a morphism of chain complexes* $F : Q \to Q'$ *such that*

$$\begin{tikzcd} Q \arrow[r, "j"] \arrow[d, "F"'] & B \arrow[d, "f"] \\ Q' \arrow[r, "j'"] & B' \end{tikzcd}$$

commutes. Moreover, any two such morphisms are chain homotopic.

PROOF. Since Q_0 is free, it is easy to see there is a homomorphism $F_0 : Q_0 \to Q'_0$ such that

$$\begin{tikzcd} Q_0 \arrow[r, "j"] \arrow[d, "F_0"'] & B \arrow[d, "f"] \\ Q'_0 \arrow[r, "j'"] & B' \end{tikzcd}$$

commutes. Since $Q_1 \to Q_0$ is a monomorphism, it is easy to see that the restriction F_1 of F_0 to Q_1 has image in Q'_1. (Since Q_1 is free, you could define F_1 even if $Q_1 \to Q_0$ were not a monomorphism.)

Suppose F, F' are two such chain morphisms. Since $j' \circ (F_0 - F'_0) = 0$, it follows from the freeness of Q_0, that there is a homomorphism $L_0 : Q_0 \to Q'_1$ such that $i' \circ L_0 = F_0 - F'_0$. Let $L_i = 0$ otherwise. It is easy to see $L : Q \to Q'$ is a chain homotopy of F to F'. $\square$

Suppose then $f : B \to B'$ and $F : Q \to Q'$ are morphisms as above. Then we obtain morphisms $\mathrm{Id} \otimes f : A \otimes B \to A \otimes B'$ and $\mathrm{Id} \otimes F : A \otimes Q \to A \otimes Q'$ consistent with it. The latter morphism induces a homomorphism $H_*(A \otimes Q) \to H_*(A \otimes Q')$. Replacing F by F' will also yield a chain homotopic morphism $\mathrm{Id} \otimes F'$, so the morphism in homology will be the same. Note that the morphism $H_0(A \otimes Q) \to H_0(A \otimes Q')$ corresponds to $\mathrm{Id} \otimes f : A \otimes B \to A \otimes B'$. However it is $H_1(A \otimes Q)$, and the morphism $H_1(A \otimes Q')$, we want to think about now.

First note that one consequence of the above lemma is that up to isomorphism $H_1(A \otimes Q)$ depends only on B (and of course also on A). For if we choose two different presentations $j : Q \to B$ and $j' : Q' \to B$, then there are chain morphisms between Q and Q'

It is clear that both compositions of these morphisms are chain homotopic to the identity, so tensoring with A and taking homology gives the desired conclusion. Now for each B, choose one specific presentation. (It could be for example formed by letting Q_0 be the free abelian group on the elements of B as basis with the obvious epimorphism $Q_0 \to B$, and Q_1 the kernel of that epimorphism. However, if B is finitely generated, you might want to restrict attention to finitely generated Q.) Define $\mathrm{Tor}(A, B) = H_1(A \otimes Q)$ for this specific presentation. For any other presentation $Q' \to B$, we have an isomorphism $\mathrm{Tor}(A, B) \to H_1(A \otimes Q')$ as above.

Another consequence of the above lemma is that $\mathrm{Tor}(A, B)$ is a functor in B. We leave it to the student to check that. $\mathrm{Tor}(A, B)$ is also a functor in A in the obvious way. □

Warning. The arguments we employed above give short shrift to some tricky issues. Namely, we rely on certain naturality properities of the objects being studied without going into detail as to what those properties are and how they relate to particular points. Hence, the student should treat this development as a preliminary treatment to be done more thoroughly in a later course devoted specifically to homological algebra.

With the above definition, we may derive the remaining properties of the Tor functor.

Proposition 9.7. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.

Proof. Let $P \to A$ be a free presentation of A and $Q \to B$ a free presentation of B as above. Then $P \otimes Q$ is a chain complex also. We shall show that $H_1(P \otimes Q) \cong H_1(A \otimes Q) \cong \mathrm{Tor}(A, B)$ and similarly $H_1(P \otimes Q) \cong H_1(P \otimes B) \cong H_1(B \otimes P) \cong \mathrm{Tor}(B, A)$.

The relations between the complexes $P\otimes Q$, $A\otimes Q$, and $P\otimes B$ and $A\otimes B$ are given by the following diagram with exact rows and columns

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
& P_1\otimes B \arrow[u] \arrow[r] & P_0\otimes B \arrow[u] \arrow[r] & A\otimes B \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_1\otimes Q_0 \arrow[u] \arrow[r] & P_0\otimes Q_0 \arrow[u] \arrow[r] & A\otimes Q_0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P_1\otimes Q_1 \arrow[u] \arrow[r] & P_0\otimes Q_1 \arrow[u] \arrow[r] & A\otimes Q_1 \arrow[u] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The two vertical homomorphisms on the left yield a morphism $P\otimes Q \to P\otimes B$ of chain complexes, and some diagram chasing shows that it induces an isomorphism of H_1. □

PROPOSITION 9.8. *Let A be an abelian group. If A or B is torsion free, then* $\mathrm{Tor}(A,B)=0$.

PROOF. We need only prove it for A by the commutativity of Tor.
If $Q\to B$ is a presentation, then

$$A\otimes Q_1 \to A\otimes Q_0$$

is an injection and $H_1(A\otimes Q)=0$. □

PROPOSITION 9.9. Tor *preserves direct sums.*

PROOF. We need only prove it for A as above.
Let $A=A'\oplus A''$. We have as chain complexes $A\otimes Q\cong A'\otimes Q\oplus A''\otimes Q$. □

PROPOSITION 9.10. *Let $0\to A'\to A\to A''\to 0$ be a short exact sequence. Then there is a natural connecting homomoprhism* $\mathrm{Tor}(A'',B)\to A'\otimes B$ *such that*

$$0\to \mathrm{Tor}(A',B)\to \mathrm{Tor}(A,B)\to \mathrm{Tor}(A'',B)\to A'\otimes B\to A\otimes B\to A''\otimes B\to 0$$

is exact. Similarly for the roles of the arguments reversed.

Proof. Again the commutativity of $\otimes$ and of Tor allows us to just prove the first assertion.

Let $Q \to B$ be a free presentation. Since Q is free,

$$0 \to A' \otimes Q \to A \otimes Q \to A'' \otimes Q \to 0$$

is an exact sequence of chain complexes. Hence, since each of these complexes has trivial homology for $k \neq 0, 1$, we get a long exact sequence

$$\begin{aligned} 0 \to H_1(A' \otimes Q) \to H_1(A \otimes Q) \to H_1(A'' \otimes Q) \\ \to H_0(A' \otimes Q) \to H_0(A \otimes Q) \to H_0(A'' \otimes Q) \to 0. \end{aligned}$$

Now replace H_0 by the tensor product and H_1 by Tor using suitable isomorphisms. □

Note that this is the only place in the development where we use the fact that Q_1 is free.

Note. Because any finitely generated abelian group is a direct sum of cyclic groups, the above results allow us to calculate $\mathrm{Tor}(A, B)$ for any finitely generated abelian groups if we do so for cyclic groups. Use of the short exact sequence $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ allows us to conclude

$$\mathrm{Tor}(\mathbf{Z}/n\mathbf{Z}, B) \cong {}_nB = \{b \in B \mid nb = 0\}.$$

From this, it is not hard to see that

$$\mathrm{Tor}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}) \cong \mathbf{Z}/\gcd(n, m)\mathbf{Z}.$$

Theorem 9.11. *Let C' and C'' be free chain complexes. Then there are natural short exact sequences*

$$0 \to \bigoplus_{r+s=n} H_r(C') \otimes H_s(C'') \xrightarrow{\times} H_n(C' \otimes C'') \to \bigoplus_{r+s=n} \mathrm{Tor}(H_r(C'), H_{s-1}(C'')) \to 0.$$

Moreover, each of these sequences splits, but not naturally.

Proof. We saw previously that $H_{r+s}(C' \otimes C'')$ fits in a short exact sequence between sums of terms made up from the cokernels of the homomorphisms

$$\mathrm{Id} \otimes i_s : H_r(C') \otimes B_s(C'') \to H_r(C') \otimes Z_s(C'')$$

and the kernels of the homomorphisms

$$\mathrm{Id} \otimes i_{s-1} : H_r(C') \otimes B_{s-1}(C'') \to H_r(C') \otimes Z_{s-1}(C'').$$

For fixed r, s, the cokernel is the tensor product, and since

$$0 \to B_{s-1} \to Z_{s-1} \to H_{s-1} \to 0$$

is a free presentation of H_{s-1}, it follows that the kernel is the required Tor term.

To establish the splitting, argue as follows. Since $B(C')$ is free, the sequence

$$0 \to Z_r(C') \to C'_r \to B_{r-1}(C') \to 0$$

splits. Thus, there is a retraction $p'_r : C'_r \to Z_r(C')$. Similarly, there is a retraction $p''_s : C''_s \to Z_s(C'')$. Define $\overline{p} : C' \otimes C'' \to H_*(C') \otimes H_*(C'')$ by

$$\overline{p}(c' \otimes c'') = \overline{p'(c')} \otimes \overline{p''(c'')}.$$

Since $p'(\partial' c') = \partial' c'$ and $p''(\partial'' c'') = \partial'' c''$—because both are cycles—it follows that $\overline{p}$ takes boundaries in $C' \otimes C''$ to zero. Restrict $\overline{p}$ to $Z(C' \otimes C'')$. This yields a homomophism $H_*(C' \otimes C'') \to H_*(C') \otimes H_*(C'')$ which by direct calculation is seen to be a retraction of $\times$. $\square$

Note. Except of the splitting, the Künneth Theorem is in fact true if either C' or C'' consists of torsion free components. Refer to any book on homological algebra, e.g., Hilton and Stammbach, for a proof.

4. Tensor and Tor for Other Rings

In algebra courses, you will study the tensor product over an arbitrary ring. For the case of a commutative ring K, we have the following additional relation in the tensor product $M \otimes_K N$ of two K-modules M, N.

$$rx \otimes y = x \otimes ry \qquad r \in K, x \in M, y \in N$$

For $K = \mathbf{Z}$, this condition follows from the biadditivity conditions. In the general case, these conditions, together with biadditivity are called *bilinearity*. The universal mapping property of the tensor product then holds for *bilinear* functions into an arbitrary K-module.

The theory of Tor_K is developed analagously for modules over a ring K, but it is much more involved. In the special case that K is a (commutative) principal ideal domain, then the theory proceeds exactly as in the case of $\mathbf{Z}$. (That is because every submodule of a free K-module is free for such rings.)

Similarly, we may define the concept of a chain complex C over a ring K by requiring all the components to be K-modules and the boundary homomorphisms to be K-homomorphisms. Also, we may define the tensor product $C' \otimes_K C''$ of two such complexes. If K is a principal ideal domain, the Künneth Theorem remains true, except that we need to put K as a subscript on $\otimes$ and Tor.

The most important case is that in which the ring K is a field. In this case, every module is free so $\mathrm{Tor}_K(M, N) = 0$ for all K-modules, i.e., vector spaces, M, N. Thus the Künneth Theorem takes the specially simple form

THEOREM 9.12. *Let C' and C'' be chain complexes over a field K. Then $\times$ provides an isomorphism*

$$H_*(C' \otimes_K C'') \cong H_*(C') \otimes_K H_*(C'').$$

The fields used most commonly in algebraic topology are $\mathbf{Q}$ (used already in the Lefshetz Fixed Point Theorem), $\mathbf{R}$, and the finite prime fields $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. Note that if A, B are vector spaces over $\mathbf{F}_p$ (which is the same as saying they are abelians groups with every non-zero element of order p), then $A \otimes_{\mathbf{F}_p} B \cong A \otimes_{\mathbf{Z}} B$. (Exercise.) However, $\mathrm{Tor}_{\mathbf{F}_p}(A, B) = 0$ while $\mathrm{Tor}_{\mathbf{Z}}(A, B) \neq 0$.

5. Homology with Coefficients

Before continuing with our study of the homology of products of spaces, we discuss a related matter for which the homological algebra is a special case of the development in the previous sections.

Recall that when discussing the Lefshetz Fixed Point Theorem, we considered the chain complex $C_*(K) \otimes \mathbf{Q}$. This is part of a more general concept. We shall discuss it in the context of singular theory, but it could be done also for simplicial or cellular theory. Let X be a space and A any abelian group. Define

$$H_*(X; A) = H_*(S_*(X) \otimes A).$$

This is called the homology of X *with coefficients in* A. Homology with coefficients has may advantages. Thus, if A has some additional structure, then that structure can usually be carried through to the homology with coefficeints in A. For example, suppose K is a field, then $S_*(X) \otimes K$ becomes a vector space under the action $a(\sigma \otimes b) = \sigma \otimes ab$. It is easy to see that the boundary homomorphism is a linear homomorphism, and it follows that $H_*(X; K)$ is also a vector space over K. (If K is a ring other than a field, the same analysis works, but we use the term 'module' instead of 'vector space'.) We made use of this idea implicitly in the case of simplicial homology when discussing the Lefshetz Theorem, because we could take traces and use other tools available for vector spaces.

5.1. The Universal Coefficient Theorem for Chain Complexes. We can do the same thing for any chain complex. Define $H_*(C; A) = H_*(C \otimes A)$. We have the following 'special case' of the Künneth Theorem.

THEOREM 9.13. *Let C be a free chain complex and A an abelian group. There are natural short exact sequences*

$$0 \to H_n(C) \otimes A \xrightarrow{i_n} H_n(C; A) \to \operatorname{Tor}(H_{n-1}(C), A) \to 0.$$

i_n is defined by $\overline{z} \otimes a \mapsto \overline{z \otimes a}$. These sequences split but not naturally.

Note. As in the case of the Künneth Theorem, this result is true in somewhat broader circumstances. See a book or course on homological algebra for details.

PROOF. It would seem that we could deduce this directly from the Künneth Theorem since we can treat A as a chain complex which is zero except in degree 0. Unfortunately, our statement of the Künneth Theorem did not have suffiently general hypotheses for this to hold. However, we can just use the same proof as follows. Let $0 \to P_1 \to P_0 \xrightarrow{j} A \to 0$ be a free presentation of A. Then

$$0 \to C \otimes P_0 \to C \otimes P_1 \to C \otimes A \to 0$$

is exact since C is free, and we get a long exact sequence

$$H_n(C \otimes P_0) \to H_n(C \otimes P_1) \to H_n(C \otimes A) \to H_{n-1}(C \otimes P_0) \to H_{n-1}(C \otimes P_1).$$

This puts $H_n(C \otimes A)$ in a short exact sequence with the cokernel of the left morphisms on the left and the kernel of the right morphism on the right. This is exactly the same situation as before, so we get exactly the same result.

The proof of splitting is done as before by using a retraction $C \to Z(C)$. □

COROLLARY 9.14. *Let X be a topological space, and A an abelian group. Then there is a natural short exact sequence*

$$0 \to H_*(X) \otimes A \to H_*(X; A) \to \operatorname{Tor}(H_*(X), A) \to 0$$

which splits but not naturally.

EXAMPLE 9.15. Let $X = \mathbf{R}P^n$. Then for each $k \le n$, we have

$$H_k(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) \cong H_k(\mathbf{R}P^n) \otimes \mathbf{Z}/2\mathbf{Z} \oplus \operatorname{Tor}(H_{k-1}(\mathbf{R}P^n), \mathbf{Z}/2\mathbf{Z}).$$

Since $H_k(\mathbf{R}P^n)$ is zero for $k > 0$ even and either $\mathbf{Z}/2\mathbf{Z}$ for k odd, and since $\operatorname{Tor}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$, it follows that

$$H_k(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z} \qquad k = 1, \dots, n.$$

We may also study the universal coefficient theorem for chain complexes over an arbitrary commutative ring K. As above the most interesting case is that of a PID, and the most interesting subcase is that of a field. The univeral coefficient theorem reads as before. The results are as above except that we must use $\otimes_K$ and Tor_K. Also if we work over a field k, then $\mathrm{Tor}_K = 0$.

6. The Eilenberg-Zilber Theorem

It might be worth your while at this point to go back and review some of the basic definitions and notation for singular homology. Remember in particular that $[p_0, \dots, p_n]$ denoted the affine map of Δ^n in some Euclidean space sending $\mathbf{e}_i$ to p_i.

Let X, Y be spaces. As noted previously, we shall show that $S_*(X \times Y)$ and $S_*(X) \otimes S_*(Y)$ are chain homotopy equivalent. To this end, we need to define chain morphisms in both directions between them. In fact we can show the existence of such morphisms with the right properties by an abstract approach called the *method of acyclic models*. In so doing we don't actually have to write down any explicit formulas. However, we can define an explicit morphism

$$A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$$

fairly readily, and this will help us do some explicit calculations later.

First, note that any singular n-simplex $\sigma : \Delta^n \to X \times Y$ is completely specified by giving its component maps $\alpha : \Delta^n \to X$ and $\beta : \Delta^n \to Y$. We abbreviate this $\sigma = \alpha \times \beta$. Define

$$A(\sigma) = \sum_{r+s=n} \alpha \circ [\mathbf{e}_0, \dots, \mathbf{e}_r] \otimes \beta \circ [\mathbf{e}_r, \dots, \mathbf{e}_n].$$

$[\mathbf{e}_0, \dots, \mathbf{e}_r]$ is an affine r-simplex called the front r-face of Δ^n, and $[\mathbf{e}_r, \dots, \mathbf{e}_n]$ is an affine s-simplex called the back s-face of Δ^n. The morphism A is called the *Alexander–Whitney map.*

Proposition 9.16. *A is a chain morphism. Moreover, it is natural with respect to maps $X \to X'$ and $Y \to Y'$ of both arguments.*

Proof. The second assertion is fairly clear from the fomula and the diagram

The first assertion is left as an exercise for the student. $\square$

To get a morphism $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ we shall use a more abstract approach.

6.1. Acyclic Models. In some of the arguments we used earlier in this course, we defined chain maps inductively. A general algebraic result which often allows us to make such constructions is the following.

PROPOSITION 9.17. *Let C, C' be two (non-negative) chain complexes with C $\mathbf{Z}$-free and C' acyclic, i.e., $H_n(C') = 0$ for $n > 0$. Let $\phi_0 : H_0(C) \to H_0(C')$ be a homomorphism.*

(i) There is a chain morphism $\Phi : C \to C'$ such that $H_0(\Phi) = \phi_0$.

(ii) Any two such chain morphisms are chain homotopic.

COROLLARY 9.18. *If C, C' are free acyclic chain complexes with $H_0(C) \cong H_0(C')$, then C, C' are chain homotopy equivalent.*

PROOF. The proof is encapsulated in the following diagrams.

$\square$

Note on the Proof. Note that by the inductive nature of the proof, we may assume that $\Phi_i, 0 \leq i < n$, with the desired property for Φ_0, have already been specified, and then we may continue defining Φ_i for $i \geq n$. Similarly, if Φ, Φ' are two such chain morphisms, and a partial chain homotopy has been specified between them, we may extend it.

Note that the above results would suffice to show that $S_*(X \times Y)$ is chain homotopy equivalent to $S_*(X) \otimes S_*(Y)$ for acyclic spaces. How to extend this to arbitrary spaces is the purpose of the theory of 'acyclic models'. The idea is to define the desired morphisms for acylic spaces inductively and the extend them to arbitrary spaces by naturality. However, the exact order in which the definitions are made is crucial. Also, the theory is stated in very great generality to be sure it applies in enough interesting cases. Hence, we need some general categorical 'nonsense' in order to proceed.

Let $\mathcal{T}$ and $\mathcal{A}$ be categories, and let $F, G : \mathcal{T} \to \mathcal{A}$ be functors. A *natural transformation* $\phi : F \to G$ is a collection of morphisms $\phi_X : F(X) \to G(X)$ in $\mathcal{A}$, one for each object X in $\mathcal{T}$ such that for each morphism $f : X \to Y$ in $\mathcal{T}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\phi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\phi_Y"] & G(Y) \end{tikzcd}$$

commutes.

This is a formalization of the concept of 'natural homomorphism' or 'natural map', so it should be familiar to you. You should go back and examine the use of this term before. Sometimes the categories and functors are a bit tricky to identify.

It is clear how to define the composition of two natural transformations. The identity morphisms $F(X) \to F(X)$ provide a natural transformation of any functor to itself. A natural transformation is called a natural equivalence if it has an inverse in the obvious sense.

We now want to set up the context for the acyclic models theorem. To understand this context concentrate on the example of the category of topological spaces and the functor $S_*(X)$ to the category $\mathcal{C}h$ of chain complexes. (Recall that by convention a chain complex for us is zero in negative degrees.)

We assume there is given a set $\mathcal{M}$ of objects of $\mathcal{T}$ which we shall call *models*. A functor $F : \mathcal{T} \to \mathcal{C}h$ is said to be *acyclic with respect to* $\mathcal{M}$ if the chain complex $F(M)$ is acyclic for each M in $\mathcal{M}$. (That means $H_i(F(M)) = 0$ for $i \neq 0$.) Such a functor is called *free in degree n with respect to* $\mathcal{M}$ if there is given an indexed subset $\mathcal{M}^n = \{M_j\}_{j \in J_n}$ of models and elements $i_j^n \in F_n(M_j)$ for each $j \in J_n$, such that for every X, an object of $\mathcal{T}$, $F_n(X)$ is free on the basis consisting of the elements

$$\{F_n(f)(i_j^n) | \quad f \in \mathrm{Hom}_{\mathcal{T}}(M_j, X),\, j \in J_n\}$$

(which are assumed to be distinct). The functor F is called *free with respect to* $\mathcal{M}$ if it is free in each degree $n \geq 0$.

Example 9.19. Let $\mathcal{T}$ be the category of $\mathcal{T}op$ of topological spaces and continuous maps. Let $\mathcal{M}$ be the collection of standard simplices Δ^n. Then $S_*(-)$ is acyclic because $S_*(\Delta^n)$ is acylic for every standard simplex. It is also free. For let $\mathcal{M}^n = \{\Delta^n\}$ and let $i^n = \mathrm{Id} : \Delta^n \to \Delta^n$. Then $S_*(\sigma)(i^n) = \sigma$ is a singular n-simplex, and $S_n(X)$ is free on the set of singular simplices.

Example 9.20. Let $\mathcal{T}$ be the category of pairs (X, Y) of topological spaces and pairs of maps (f, g). (We don't assume $Y \subseteq X$. The components may be totally unrelated.) Let the model set be the set of pairs (Δ^r, Δ^s) of standard simplices. Consider first the the functor $S_*(X \times Y)$. This is acyclic because each of the spaces $\Delta^r \times \Delta^s$ is acylic. It is also free. For let $\mathcal{M}^n = \{(\Delta^n, \Delta^n)\}$ and let $i^{n,n}$ be the *diagonal* map $\Delta^n \to \Delta^n \times \Delta^n$ defined by $x \mapsto (x, x)$. Let $\sigma = \alpha \times \beta$ be a singular n-simplex in $X \times Y$. Then $\sigma = S_n(\alpha, \beta)(i^{n,n})$ so $S_n(X \times Y)$ is free as required.

Consider next the functor $S_*(-) \otimes S_*(-)$. This is acyclic because for standard simplices $H_i(\Delta^r) = 0$ for $r \neq 0$ and similarly for $H_j(\Delta^s) = 0$, and these are both $\mathbf{Z}$ in degree zero. We may now apply the Künneth Theorem for chain complexes to conclude that $S_*(\Delta^r) \otimes S_*(\Delta^s)$ is acyclic. This functor is also free. To see this, fix an $n \geq 0$. Let $\mathcal{M}^n = \{(\Delta^r, \Delta^s) \mid r + s = n\}$, and let $i^{r,s} = i^r \otimes i^s$. Then $F_n(X, Y) = \bigoplus_{r+s=n} S_r(X) \otimes S_s(Y)$ is free on the basis consisting of all $\alpha \otimes \beta$ where α is a singular r-simplex in X, β is an singular s- simplex in Y, and $r + s = n$. However, $\alpha \otimes \beta = S_r(\alpha) \otimes S_s(\beta)(i^{r,s})$.

THEOREM 9.21. *Let $\mathcal{T}$ be a category with a set of models $\mathcal{M}$. Let F and G be functors to the category of (non-negative) chain complexes. Suppose F is free with respect to $\mathcal{M}$ and G is acyclic with respect to $\mathcal{M}$. Suppose there is given a natural transformation of functors $\phi_0 : H_0 \circ F \to H_0 \circ G$. Then there is a natural transformation of functors $\Phi : F \to G$ which induces ϕ_0 is homology in degree zero. Moreover, any two such natural transformations Φ, Φ' are chain homotopic by a natural chain homotopy. That is, there exist natural transformations $D_n : F_n \to G_{n+1}$ such that for each X in $\mathcal{T}$, we have*

$$\Phi_{n,X} - \Phi'_{n,X} = \partial_{n+1}^{G(X)} \circ D_{n,X} + D_{n-1,X} \circ \partial_n^{F(X)}.$$

COROLLARY 9.22. *Let $\mathcal{T}$ be a category with models $\mathcal{M}$. Let F and G be functors to the category of (non-negative) chain complexes which are both acyclic an free with respect to $\mathcal{M}$. If $H_0 \circ F$ is naturally equivalent to $H_0 \circ G$, then F is naturally chain homotopically equivalent to G.*

We leave it to the student to work out exactly what the Corollary means and to prove it.

PROOF. By hypothesis, for each model M in $\mathcal{M}$, $F(M)$ is a free chain complex and $G(M)$ is acyclic. Also, $\phi_{0,M} : H_0(F(M)) \to H_0(G(M))$ is given. By the previous proposition, there is a chain morphism $\psi_M : F(M) \to G(M)$ which induces ϕ_0 in degree zero homology. For X in $\mathcal{T}$, $M_j, j \in J_0$, a model of degree 0, $f \in \mathrm{Hom}_{\mathcal{T}}(M_j, X)$, define

$$\Phi_{0,X}(F(f)(i_j^0)) = G_0(f)(\psi_{0,M_j}(i_j^0)) \in G_0(X).$$

This specifies $\Phi_{0,X}$ on a basis for $F_0(X)$, so it yields a homomophism

$$\Phi_{0,X} : F_0(X) \to G_0(X).$$

It is not hard to see that the collection of these homomorphisms is a natural transformation $\Phi_0 : F_0 \to G_0$. It is also true that each of the diagrams

$$\begin{tikzcd}
F_0(X) \arrow[r, "\Phi_{0,X}"] \arrow[d] & G_0(X) \arrow[d] \\
H_0(F(X)) \arrow[r, "\phi_{0,X}"] & H_0(G(X))
\end{tikzcd}$$

commutes. To see this note that for each M_j of degree 0, and each $f : M_j \to X$ in $\mathcal{T}$, there is a cubical diagram:

$$\begin{tikzcd}
F_0(M_j) \arrow[rr, "\psi_{0,M_j}"] \arrow[dr] \arrow[dd] & & G_0(M_j) \arrow[dr] \arrow[dd] & \\
& F_0(X) \arrow[rr, "\Phi_{0,X}"] \arrow[dd] & & G_0(X) \arrow[dd] \\
H_0(F(M_j)) \arrow[rr, "\phi_{0,M_j}"] \arrow[dr] & & H_0(G(M_j)) \arrow[dr] & \\
& H_0(F(X)) \arrow[rr, "\phi_{0,X}"] & & H_0(G(X))
\end{tikzcd}$$

The back face commutes by the defining property of the top arrow. The two side faces of this cube commute because the morphism from cycles to homology is a natural transformation of functors on chain complexes. The bottom face commutes because ϕ_0 is a natural transformation of functors. The top face is not necessarily commutative but for the element i_j^0, it commutes by the definition of $\Phi_{0,X}$. This establishes that the front face commutes on a basis for the upper left corner $F_0(X)$, so it commutes.

Now suppose inductively that natural transformations $\Phi_i : F_i \to G_i$ have been defined which commute with the boundary homomorphisms for $0 \leq i < n$. Assume also that the morphisms $\psi_M : F(M) \to G(M)$ have been modified so that $\psi_{i,M_j} = \Phi_{i,M_j}, 0 \leq i < n$ for each model M_j

of degree n. For such an M_j and $f : M_j \to X$ in $\mathcal{T}$, define

$$\Phi_{n,X}(F_n(f)(i_j^n)) = G_n(f)(\psi_{n,M_j}(i_j^n))$$

and extend by linearity to get a homomorphism

$$\Phi_{n,X} : F_n(X) \to G_n(X).$$

As above, this defines a natural transformation of functors. Also,

$$\begin{tikzcd} F_n(X) \arrow[r, "\Phi_{n,X}"] \arrow[d, "\partial_n"'] & G_n(X) \arrow[d, "\partial_n"] \\ F_{n-1}(X) \arrow[r, "\Phi_{n-1,X}"] & G_{n-1}(X) \end{tikzcd}$$

commutes by a three dimensional diagram chase as above. We leave the details to the student.

The arguments for chain homotopies are done essentially the same way. If $\Phi, \Phi' : F \to G$ both induce ϕ_0 in degree 0, then by the proposition there is a chain homotopy from Φ_M to Φ'_M for each model M. Using the value of this chain homotopy on i_j^0 for M_j, a model of degree 0, we may define a natural transformation $D_0 : F_0 \to G_1$ with the right property. This may then be lifted inductively to $D_n : F_n \to G_{n+1}$ as above. We leave the details to the student.

□

THEOREM 9.23. *The Alexander–Whitney chain morphism*

$$A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$$

is a chain homotopy equivalence. In particular, there is a natural chain morphism $B : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ such that $A \circ B$ and $B \circ A$ are both naturally chain homotopic to the respective identities.

PROOF. This follows directly from the acyclic models theorem with the categories and models specified in Example 9.20. In degree 0, the Alexander–Whitney morphism sends $\alpha \times \beta$ to $\alpha \otimes \beta$, and it is an isomorphism

$$S_0(X \times Y) \cong S_0(X) \otimes S_0(Y).$$

This in turn yields the natural isomorphism $H_0(S_*(X\times Y)) \to H_0(S_*(X) \otimes S_*(Y)) \cong H_0(S_*(X)) \otimes H_0(S_*(Y))$. □

THEOREM 9.24 (Künneth Theorem for Singular Homology). *Let X, Y be spaces. There are natural short exact sequences*

$$0 \to \bigoplus_{r+s=n} H_r(X)\otimes H_s(Y) \xrightarrow{\times} H_n(X\times Y) \to \bigoplus_{r+s=n-1} \mathrm{Tor}(H_r(X), H_s(Y)) \to 0.$$

These sequences split but not naturally.

EXAMPLE 9.25. Let $X = Y = \mathbf{R}P^2$. Then

$$H_0(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathbf{Z}$$
$$H_1(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathbf{Z} \otimes \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \otimes \mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$$
$$H_2(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathbf{Z}/2\mathbf{Z} \otimes \mathbf{Z}/2\mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z}$$
$$H_3(\mathbf{R}P^2 \times \mathbf{R}P^2) \cong \mathrm{Tor}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}.$$

Note that each of these isomorphisms is actually natural because in each case one term in the non-natural direct sum is trivial.

The acyclic models theorem works just as well for functors into chain complexes over a PID. In particular, if K is a field, we get the following stronger result.

THEOREM 9.26. *Let X, Y be spaces, and let K be a field. Then $\times$ provides a natural isomorphism*

$$H_*(X; K) \otimes_K H_*(Y; K) \cong H_*(X \times Y; K)$$

Note that although the chain complex Künneth Theorem is relatively simple in this case, working over a field does not materially simplify the Eilenberg–Zilber Theorem.

CHAPTER 10

Cohomology

1. Cohomology

As you learned in linear algebra, it is often useful to consider the *dual* objects to objects under consideration. This principle applies much more generally. For example, in order to understand a differentiable manifold or algebraic variety, it is useful to study the appropriate functions on it. In algebraic topology, a similar idea is frutiful. Instead of considering chains which are linear combinations of simplices or cells, we consider functions on simplices or cells. The resulting theory is called *co*homology, and it is dual to homology. For us the most important aspect of cohomology theory is that under appropriate circumstances it gives us a *ring* structure. This allows for more use of algebraic techniques in solving geometric problems.

Let X be a space, and let M be an abelian group. Denote by $S^n(X;M)$ the set of all functions defined on the set of singular n-simplices of X with values in M. Any such function defines a unique homomorphism $f : S_n(X) \to M$ and conversely any such homomorphism defines such a function. Hence, we may also write

$$S^n(X;M) = \mathrm{Hom}(S_n(X), M).$$

We shall show how to make S^n into the analogue of a chain complex, but it is worthwhile doing that in a somewhat more general context.

1.1. Some Homological Algebra. A *cochain complex* C^* is a collection of abelian groups $C^n, n \in \mathbf{Z}$ and homomorphisms $\delta^n : C^n \to C^{n+1}$ such that $\delta^{n+1} \circ \delta^n = 0$ for each n. A cochain complex is called non-negative if $C^n = \{0\}$ for $n < 0$. We shall consider only non-negative cochain complexes unless otherwise states.

The easiest way to construct a cochain complex is from a chain complex. Namely, let S_* denote a chain complex, and let M be an abelian group. Let

$$C^n = \mathrm{Hom}(S_n, M),$$

and let $\delta^n = \mathrm{Hom}(\partial_{n+1}, \mathrm{Id})$, i.e.,

$$\delta^n(f) = f \circ \partial_{n+1} \qquad f \in \mathrm{Hom}(S_n, M).$$

Here $\mathrm{Hom}(M, N)$ denotes the set of homomorphisms from one abelian group into another. This is the object part of a functor into the category of abelian groups. Given, $i : M \to M', j : N \to N'$, we also have the homomophism $\mathrm{Hom}(i, j) : \mathrm{Hom}(M', N) \to \mathrm{Hom}(M, N')$ defined by

$$\mathrm{Hom}(i, j)(f) = j \circ f \circ i \qquad f \in \mathrm{Hom}(M', N).$$

Note that his functor is *contravariant* in the first variable, and covariant in the second, i.e.,

$$\mathrm{Hom}(i_1 \circ i_2, j_1 \circ j_2) = \mathrm{Hom}(i_2, j_1) \circ \mathrm{Hom}(i_1, j_2).$$

(You should draw some diagrams and check for yourself that everything makes sense and that the rule is correct.)

The functor Hom preserves finite direct sums, i.e.,

$$\mathrm{Hom}(\bigoplus_i M_i, N) \cong \bigoplus_i \mathrm{Hom}(M_i, N)$$

$$\mathrm{Hom}(M_i, N) \cong \bigoplus_i \mathrm{Hom}(M, N_i).$$

The second statement in fact holds for any direct sums, but the first statement only holds for finite direct sums. The first statement follows from the fact that any homomorphism $f : \bigoplus_i M_i \to N$ is completely determined by its restrictions f_i to the summands M_i. Moreover, the correspondence

$$f \leftrightarrow (f_i)_i$$

actually provides an isomorphism of abelian groups. Similar remarks apply to the second isomorphism.

Another important fact about Hom is the relation

$$\mathrm{Hom}(\mathbf{Z}, N) \cong N$$

the isomorphism being provided by $f \in \mathrm{Hom}(Z, N) \mapsto f(1)$ and $a \in N \mapsto f_a$ where $f_a(n) = na$. (This is analogous to the isomorphism $M \otimes \mathbf{Z} \cong M$.)

The functor Hom is *left exact*. That means the following

Proposition 10.1. *Let $0 \to M' \to M \to M'' \to 0$ be a ses of abelians groups and let N be an abelian group. Then*

$$0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N)$$

is exact. Similarly, if $0 \to N' \to N \to N'' \to 0$ is exact then

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M.N) \to \mathrm{Hom}(M, N'')$$

is exact.

In fact, in these assertions, we may drop the assumption that the 'M' sequence is exact on the left or that the 'N' sequence is exact on the right.

Note the reversal of arrows in the first statement. In effect this says that Hom preserves kernels for the second variable and sends cohernels to kernels for the first variable.

PROOF. You just have to check what each of the assertions means. We leave it as an exercise for the student. □

It is easy to see that Hom is not generally exact. For example, $0 \to Z \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$ and $N = \mathbf{Z}$ yields

$$0 \to \mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}) = 0 \to \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) = \mathbf{Z} \xrightarrow{2} \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) = \mathbf{Z}$$

and the last homomophism is not onto. We shall return to the question of when Hom is exact later.

Returning to our study of cochain complex, note that $C^* = \mathrm{Hom}(S_*, M)$ is a cochain complex because

$$\delta^{n+1} \circ \delta^n = \mathrm{Hom}(\partial_{n+2}, \mathrm{Id}) \circ \mathrm{Hom}(\partial_{n+1}, \mathrm{Id}) = \mathrm{Hom}(\partial_{n+1} \circ \partial_{n+2}, \mathrm{Id}) = 0.$$

We may in fact sumbsume the theory of cochain complexes in that of chain complexes as follows. Given a cochain complex C^*, define $C_n = C^{-n}$ and let $\partial_n : C_n \to C_{n-1}$ be $\delta^{-n} : C^{-n} \to C^{-n+1} = C^{-(n-1)}$. Note that non-negative cochain complexes correspond exactly to non-positive chain complexes. Hence, the notions of cycles, boundaries, homology, chain homotopy, etc. all make sense for cochain complexes. However, we shall prefix everything by 'co', and use the appropriate superscript notation when discussing cochain complexes. (Note however, that all the arrows go in the 'opposite' direction when using superscript notation.) In particular, if C^* is a cochain complex, then

$$H^n(C^*) = Z^n(C^*)/B^n(C^*) = \mathrm{Ker}(\delta^n)/\,\mathrm{Im}(\delta^{n-1})$$

is the nth *cohomology* group of the complex.

As above, let X be a topological space, M an abelian group, and define

$$H^*(X; M) = H^*(\mathrm{Hom}(S_*(X), M)).$$

Because of the above remarks, it is easy to see that this is a functor both on topological spaces X and also on abelian groups M. However, it is contravariant on topological spaces. In more detail, if $f : X \to Y, g : Y \to Z$ are maps, then we have

$$S_*(g \circ f) = S_*(g) \circ S_*(f) : S_*(X) \to S_*(Z)$$

and

$$S^*(g \circ f, \mathrm{Id}) = \mathrm{Hom}(g \circ f, \mathrm{Id}) = \mathrm{Hom}(f, \mathrm{Id}) \circ \mathrm{Hom}(g, \mathrm{Id}) : \\ \mathrm{Hom}(S_*(Z), M) \to \mathrm{Hom}(S_*(Y), M) \to \mathrm{Hom}(S_*(X), M).$$

Hence,

$$H * (g \circ f; \mathrm{Id}) = H^*(f; \mathrm{Id}) \circ H^*(g\circ; \mathrm{Id}) : \\ H^*(Z; M) \to H^*(Y; M) \to H^*(X; M).$$

PROPOSITION 10.2. *Let X be a topological space with finitely many components, M an abelian group. Then*

$$H^0(X; M) = \bigoplus_{\textit{comps of } X} M$$

PROOF. We have by definition, an exact sequence

$$S_1(X) \xrightarrow{\partial_1} S_0(X) \to H_0(X) \to 0$$

Hence,

$$0 \to \mathrm{Hom}(H_0(X), M) \to \mathrm{Hom}(S_0(X), M) \xrightarrow{\delta^0} \mathrm{Hom}(S_1(X), M)$$

is exact. The result now follows from the fact that $H_0(X)$ is free on the components of X. □

What do you think the result should be if X has infinitely many components?

We may repeat everything we did for singular homology except that the arrows all get turned around. Thus, we define

$$\tilde{H}_0(X; M) = \mathrm{Coker}(H^0(\{P\}; M) = M \to H^0(X; M)).$$

This may also be dscribed by taking the homology of the cochain complex

$$\mathrm{Hom}(\tilde{S}_n(X), M)$$

Suppose now that A is a subspace of X. Consider the short exact sequence of of chain complexes

$$0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0.$$

Since $S_n(X, A)$ is free, it follows that the above sequence yields a *split* short exact sequence of abelian groups for each n. (However, the splitting morphisms $S_n(X, A) \to S_n(X)$ won't generaly constitute a chain map!) Since Hom preservers direct sums (and also splittings), it follows that

$$0 \to \mathrm{Hom}(S_*(X, A), M) \to \mathrm{Hom}(S_*(X), M) \to \mathrm{Hom}(S_*(A), M) \to 0$$

is an exact seqeunce of chain complexes. (It also splits for each n, but it doesn't necessarily split as a sequence of cochain complexes.) Define

$$H^*(X, A; M) = H^*(\mathrm{Hom}(S_*(X, A), M).$$

Then we get connecting homomorphisms

$$\delta^n : H^n(A; M) \to H^{n+1}(X, A; M)$$

such that

$$\to H^n(X, A; M) \to H^n(X; M) \to H^n(A; M) \to H^{n+1}(X, A; M) \to \dots$$

is exact. As in the case of homology, there is a similar sequence for reduced cohomology.

The homotopy axiom holds because the chain homotopies induce 'cochain homotopies' of the releveant cohaing complexes.

Excision holds, i.e., if $\overline{U}$ is contained in the interior of A, then $H^*(X, A; M) \to H^*(X - U, A - U; M)$ is an isomorphism. (Of course, the direction of the homomorphism is rversed.

Finally, there is a Mayer–Vietoris sequence in cohomology which looks exactly like the one in homology except that all the arrows are reversed.

Note that cohomology always requires a coefficient group. Hence, it is analagous to singular cohomology with coefficients. As in that theory, we also get a long exact sequence from coefficient sequences.

PROPOSITION 10.3. *Let $0 \to N' \to N \to N'' \to 0$ be a s.e.s. of abelian groups. There is a natural connecting homomoprhism $H^n(X; N'') \to H^{n+1}(X; N')$ such that the long sequence*

$$\to H^n(X; N') \to H^n(X; N'') \to H^n(X; N'') \to H^{n+1}(X; N') \to \dots$$

is exact. (There is also a similar sequence for reduced cohmology.)

PROOF. Apply the following lemma to obtain the s.e.s

$$0 \to \mathrm{Hom}(S_*(X), N') \to \mathrm{Hom}(S_*(X), N) \to H_*(S_*(X), N'') \to 0.$$

LEMMA 10.4. *Let M be free. Then for each ses $0 \to N' \to N \to N'' \to 0$, the sequence*

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to 0$$

is exact.

PROOF. We need only show that $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'')$ is an epimorphism. But this amounts to showing that every homomorphism $f'' : M \to N''$ may be lifted to a homomorphism $f : M \to N$.

However, this follows because $N \to N''$ is an epimorphism and M is free. □

□

2. The Universal Coefficient Theorem

One might be tempted to think that cohomology is simply dual to homology, but that is not generally the case. There is a universal coefficient theorem analagous to that for homology with coefficients.

Let C_* be a chain complex. Define a homomophism

$$\alpha : H^n(\mathrm{Hom}(C_*, N) \to \mathrm{Hom}(H_n(C_*), M)$$

by

$$\alpha(\overline{f})(\overline{z}) = f(z)$$

for f an n-cocycle in $\mathrm{Hom}(C_n, M)$ and z a cycle in C_n. We leave it to the student to check that this is well defined, i.e., it depends only on the cohomology class of f and the homology class of z.

We shall show that if C_* is free, then α is always onto, and in good cases it is an isomorphism. To this end, consider as in the case of homology the short exact sequence of chain complexes

$$0 \to Z_*(C_*) \to C_* \to B_{+,*} \to 0.$$

These split in each degree, so

$$0 \to \mathrm{Hom}(B_{+,*}, N) \to \mathrm{Hom}(C_*, N) \to \mathrm{Hom}(Z_*, N) \to 0$$

is an exact sequence of cochain complexes. (The coboundaries on either end are trivial.) Hence, this induces a long exact sequence in cohomology

$$\to \mathrm{Hom}(B_{n-1}, N) \to H^n(\mathrm{Hom}(C_*, N)) \to \mathrm{Hom}(Z_n, N) \xrightarrow{\Delta^n} \mathrm{Hom}(B_n, N) \to$$

Note the shift in degree relating $B_{+,n}$ to B_{n-1}. This yields short exact seqeunces

$$0 \to \mathrm{Coker}\, \Delta^{n-1} \to H^n(\mathrm{Hom}(C_*, N)) \to \mathrm{Ker}\, \Delta^n \to 0.$$

It is not hard to check that $\Delta^n : \mathrm{Hom}(Z_n, N) \to \mathrm{Hom}(B_n, N)$ is just dual to the inclusion $i_n : B_n \to Z_n$. From the short exact sequence

$$0 \to B_n \to Z_n \to H_n \to 0$$

and the left exactness of Hom, it follows that

$$0 \to \mathrm{Hom}(H_n, N) \to \mathrm{Hom}(Z_n, N) \to \mathrm{Hom}(B_n, N) \tag{44}$$

is exact, so we may identify $\operatorname{Ker} \Delta^n$ with $\operatorname{Hom}(H_n, N)$. Hence, we get an epimorphism

$$H^n(\operatorname{Hom}(C_n, N)) \to \operatorname{Hom}(H_n(C_*), N)$$

which it is easy to check is just the homomorphism α defined above.

It remains to determine $\operatorname{Coker} \Delta^{n-1}$. Shifting degree up by one, we see we need to find the cokernel of the homomophism on the right of 44. Of course, the best case will be when that cokernel is trivial, i.e., when the functor Hom is exact on the right as well as on the left for that particular sequence.

Consider now the general problem the above discussion suggests. Let M be an abelian group and let $0 \to P_1 \to P_0 \to M \to 0$ provide a free presentation as in our discussion of Tor. Consider the cochain complex $\operatorname{Hom}(P, N)$ and define a bifunctor $\operatorname{Ext}(M, N)$ such that

$$\operatorname{Ext}(M, N) \cong H^1(\operatorname{Hom}(P, N)) = \operatorname{Coker}\{\operatorname{Hom}(P_0, N) \to \operatorname{Hom}(P_1, N)\}.$$

The theory at this point proceeds in a completely analagous fashion to the theory of Tor with some important exceptions. The functor $\operatorname{Ext}(M, N)$ is well defined. It is contravariant in the first argument (reverses arrows) and covariant in the second argument. It preserves *finite* direct sums. It is not generally commutative, i.e., there is no particular relation between $\operatorname{Ext}(M, N)$ and $\operatorname{Ext}(N, M)$ since there is no such relation for Hom. Because of the lack of commutativity, some of the proofs which relied on commutativity in the case of Tor have to be recast.

One very important property of Ext is the following

PROPOSITION 10.5. *Let $0 \to M' \to M \to M'' \to 0$ be exact. Then there is a natural connecting homomorphism such that the sequence*

$$\begin{aligned} 0 \to \operatorname{Hom}(M'', N) \to \operatorname{Hom}(M.N) \to \operatorname{Hom}(M', N) \to \\ \operatorname{Ext}(M'', N) \to \operatorname{Ext}(M, N) \to \operatorname{Ext}(M', N) \to 0 \end{aligned}$$

is exact. There is an analagous sequence for every short exact sequence in the second variable.

We shall not go through the proofs of these results here, but the student should think about them. Refer back to the section on Tor for guidance. You will see this done in more detail in a course in homological algebra or you can look yourself in one of the standard references on reserve. If you study the appropriate homological algebra, you will also see why the name 'Ext' is used to describe the functor.

Note that the above facts allow us to compute Ext for every finitely generated abelian group. The additivity reduces the problem to that of cyclic groups. From the definition, $\mathrm{Ext}(Z, N) = 0$ since $\mathbf{Z}$ is free and 'is' its own presentation. The s.e.s $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$ yields

$$0 \to \mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, N) \to \mathrm{Hom}(\mathbf{Z}, N) \xrightarrow{n} \mathrm{Hom}(\mathbf{Z}, N) \to \mathrm{Ext}\,\mathbf{Z}/n\mathbf{Z}, N) \to 0.$$

Since $\mathrm{Hom}(Z, N) \cong N$ This shows

$$\mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, N) \cong {}_nN$$
$$\mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, N) \cong N/nN.$$

It follows that (as with Tor)

$$\mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}) \cong \mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z}) \cong \mathbf{Z}/\gcd(n, m)\mathbf{Z}.$$

To round this out, note that $\mathrm{Hom}(\mathbf{Z}, N) = N, \mathrm{Ext}(\mathbf{Z}, N) = 0, \mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}) = 0, \mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$.

The lack of symmetry of Hom, as mentioned above, prevents us from proceeding in a completely symmetrical manner, as we did for Tor. It is possible to define Ext with an appropriate construction using the second argument. What is needed is a dual concept to that of a presentation, where the group N is imbedded in an appropriate kind of group Q_0. We shall not go into this in more detail here.

You may also have noted above that $\mathrm{Ext}(M, N)$ need not vanish if N is free. However, it does vanish if N has the following property: for each $x \in N$ and each positive integer n there is a $y \in N$ such that $x = ny$. A group with this property is called *divisible*, and $\mathrm{Ext}(M, N) = 0$ if N is divisible. This is easy to derive if M is finitely generated from the formula

$$\mathrm{Ext}(\mathbf{Z}/n\mathbf{Z}, N) \cong N/nN.$$

However, if M is not finitely generated, one must use transfinite induction (Zorn's Lemma). We also leave this for you discover later when you study homological algebra in greater depth. The most interesting divisible groups are $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$. Moreover, any field of characterstic zero is divisible.

Note that this means that if N is divisible, then Hom is an exact functor.

We may now apply the above results to the free presentation

$$0 \to B_n(C_*) \to C_n \to H_n(C) \to 0$$

of homology to obtain

THEOREM 10.6. *Let C_* be a free chain complex, N an abelian group. Then there are natural short exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C_*), N) \to H^n(\mathrm{Hom}(C_*, N)) \to \mathrm{Hom}(H_n(C_*), N) \to 0$$

which splits (but not naturally).

PROOF. The splitting follows by an argument analogous to the one used for homology. □

Applying this to singular cohomology, we obtain

THEOREM 10.7. *Let X be a topological space, N an abelian group. Then there are natural short exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(X), N) \to H^n(X; N) \to \mathrm{Hom}(H_n(X), N) \to 0$$

which split (but not naturally).

Note also that the morphism on the right is just α which arises through evaluation of cocyles on cycles. Also, we could equally well have stated the result for the cohomology of a pair (X, A).

EXAMPLE 10.8. $H^k(S^n; N) \cong 0$ unless $k = 0, n$, in which case it is n.

EXAMPLE 10.9. For any space X, $H^k(X; \mathbf{Q})$ is a vector space over $\mathbf{Q}$ of the same dimension as the rank of $H_k(X)$.
(All the Ext terms are zero in these cases.)

EXAMPLE 10.10. $H^k(\mathbf{R}P^n; \mathbf{Z}) = 0$ in all odd degrees except if $k = n$ is odd, in which case we get $\mathbf{Z}$. It is generally $\mathbf{Z}/2\mathbf{Z}$ in even degrees $k > 0$ and of course $H^0(\mathbf{R}P^n; \mathbf{Z}) = \mathbf{Z}$.
On the other hand, $H^k(\mathbf{R}P^n, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ for $0 \le k \le n$.
We leave it to the student to verify these assertions and to calculate the corresponding groups for $\mathbf{C}P^n$ and $\mathbf{H}P^n$.

The above discussion of Ext works just as well for complexes which are modules over an appropriate ring K. As in the case of Tor, if K is a PID, then we get the same universal coefficient theorem, except that we need to suscript for the ring. $\mathrm{Hom}_K(M, N)$ denotes the K-module homomorphisms of $M \to N$, and Ext_K denotes the corresponding derived functor. The most interesting case is that in which K is a field. Then every K-module is free and $\mathrm{Ext}_K(M, N) = 0$ in all cases.

PROPOSITION 10.11. *Let X be a space, K a field, and N a vector space over K. Then, we have isomorphisms*

$$\alpha : H^n(X; N) \xrightarrow{\cong} \mathrm{Hom}_K(H_n(X; K), N)$$

Note that on the right hand side we have $H_n(X; K)$. The point is that the universal coefficient theorem over K refers to chain complexes defined over K, so we must apply it to $S_*(X) \otimes K$ in order to get such a chain complex.

3. Cup Products

It is unfortunately true that the algebraic invariants provided by the homology groups of a space are not adequate to tell different spaces apart. For example, it is often possible to match the homology of a space simply by taking wedges of appropriate spaces. Thus $S^{2n} \vee S^{2n-2} \vee \cdots \vee S^2$ has the same homology as $\mathbf{C}P^n$. Many similar examples abound. One trend in algebraic topology has been to supplement previously know algebraic structures with additional ones to provide greter 'resolution' in attacking geometric problems. As mentioned earlier, one advantage of *co*homology is that it may be endowed with a multiplicative structure.

Let X be a space and K a commutative ring. Common choices for K would be $K = \mathbf{Z}$ or K a field such as $\mathbf{Q}, \mathbf{R}, \mathbf{C}$, or $\mathbf{Z}/p\mathbf{Z}$ for p a prime. We shall show how to make $H^*(X; K)$ into a ring. Then it will often be the case that spaces with the same additive groups $H^k(X; K)$ in each degree may be distinguished by their multiplicative structures.

The product may be defined as follows. Let $f : S_r(X) \to K$ and $g : S_s(X) \to K$ be cochains. Define $f \cup g : S_{r+s}(X) \to K$ by

$$(f \cup g)(\sigma) = f(\sigma \circ [\mathbf{e}_0, \dots \mathbf{e}_r]) g(\sigma \circ [\mathbf{e}_r, \dots, \mathbf{e}_n]) \qquad \sigma \text{ a singular } n\text{-simplex.}$$

Using this formula, it is easy to check the following.

(i) If f, g are cocyles $f \cup g$ is also a cocyle. Also, changing f and g by coboundaries changes $f \cup g$ by a coboundary. These facts follow from the following formula which we shall leave as an exercise for the student.

$$\delta^{r+s}(f \cup g) = \delta^r f \cup g + (-1)^r f \cup \delta^s g.$$

Hence, we may define $\xi \cup \eta$ to be the class of the cocycle $f \cup g$. where f, g represent the cohomology classes $\xi \in H^r(X; K), \eta \in H^s(X; K)$ respectively.

(ii) The product so defined satisfies the appropriate distributive and associative laws. These rules in fact are already satisfied at the cochain level.

(iii) The cup product is functorial, i.e., if $\phi : X \to Y$ is a map of spaces, then $\phi^* : H^*(Y; K) \to H^*(X; K)$ preserves products, i.e.,

$$\phi^*(\xi \cup \eta) = \phi^*(xi) \cup \phi^*(\eta) \qquad \xi \in H^r(Y; K), \eta \in H^s(X; K).$$

We leave these verifications for the student. (See also the more abstract approach below.)

In general, let C^* be a cochain complex, and suppose a product structure is defined on C^* such that

(a) $C^r C^s \subseteq C^{r+s}$.

(b) The associative and distributive laws hold for the product.

(c) If $u \in C^r, v \in C^s$, then $\delta^{r+s}(uv) = (\delta^r u)v + (-1)^r u(\delta^s v)$.

Then we say that C^* is a *differential graded ring.* (It is often true that we are interested in the case where C^* is also an algebra over a field K thought of as imbedded in C^0, in which case C^* is called a differential graded algebra or 'DGA'.) Thus, the singular cochain complex of a space is a differential graded ring. In general, if C^* is a differential graded ring, then $H^*(C^*)$ becomes a *graded* ring.

There is one additional fact about the cup product that is important. It is 'commutative' in the following graded sense. If $\xi \in H^r(X;K), \eta \in H^s(X;K)$, then

$$\xi \cup \eta = (-1)^{rs} \eta \cup \xi.$$

A graded ring with this property is called a *graded commutative* ring. This rule is not so easy to derive from the formula defining the cup product for cochains, since it does not hold at the cocycle level but only holds up to a coboundary. To prove it, we shall use a more abstract approach which also gives us a better conceptual understanding of where the cup product arises. We shall show how to construct the cup product as a composite homomorphsim

$$\cup : H^*(X;K) \otimes_K H^*(X;K) \to H^*(X \times X;K) \to H^*(X;K).$$

Note that first specifying a product $H^r \times H^s \to H^{r+s}$ denoted $(u,v) \mapsto uv$ satisfies the distributive laws if and only if it is bilinear so that it induces $H^r \otimes_K H^s \to H^{r+s}$ and converses any such homomorphism induces such a product by $u \otimes v \mapsto uv$. Hence, the above is a plausible reformulation.

Secondly, the second constituent of the above homomorphism is easy to come by. Namely, let $\Delta : X \to X \times X$ and use $\Delta^* : H^*(X \times X;K) \to H^*(X)$.

The first constituent of the cup product homomorphism is a bit harder to get at. It basically comes from the Alexander–Whitney morphism. Namely, let X, Y be spaces. Then $A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ induces a morphism of cochain complexes

$$A^* : \mathrm{Hom}(S_*(X) \otimes S_*(Y), K) \to \mathrm{Hom}(S_*(X \times Y), K).$$

(Note that any other 'Eilenberg–Zilber' morphism—which would be chain homotopic to A—could be substituted here, but the Alexander–Whitney map is explicit, so it makes sense to use it.) We complement this with the morphism of cochain complexes

$$\boxtimes : \mathrm{Hom}(S_*(X);K) \otimes_K \mathrm{Hom}(S_*(Y), K) \to \mathrm{Hom}(S_*(X) \otimes S_*(Y), K)$$

defined as follows. Let $f : S_r(X) \to K, g : S_s(Y) \to K$ and define $f \boxtimes g : S_r(X) \otimes S_s(Y) \to K$ by

$$(f \boxtimes g)(\alpha \otimes \beta) = f(\alpha)g(\beta) \qquad \alpha \in S_r(X), \beta \in S_r(Y).$$

It is easy to see that the right hand side is bi-additive, so the formula defines an element of $\mathrm{Hom}(S_r(X) \otimes S_s(Y), K)$. It is also easy to see that $(f, g) \mapsto f \boxtimes g$ is bi-additive in f and g respectively. However, you will note that on the left hand side the tensor product sign has K as a subscript. That means we are treating $\mathrm{Hom}(S_*, K)$ as a module over K and then taking the tensor product as K-modules. The K- module structure is defined by $(af)(\alpha) = a(f(\alpha)$ where $a \in K, f \in \mathrm{Hom}(S_*, K)$ and $\alpha \in S_*$. That $f \boxtimes g$ is bilinear follows from

$$\begin{aligned}((af) \boxtimes g)(\alpha \otimes \beta) &= (af)(\alpha)g(\beta) = a(f(\alpha))g(\beta) \\ &= f(\alpha)(a(g(\beta)) = f(\alpha)(ag)(\beta) = (f \boxtimes (ag))(\alpha \otimes \beta).\end{aligned}$$

We shall denote the composite morphism $A^*(f \boxtimes g)$ by $f \times g$ and similarly for the induced homomorphism

$$\times : H^*(X; K) \otimes H^*(Y; K) \to H^*(X \times Y; R).$$

We now define the cup product by the rather cumbersome formula

$$f \cup g = \Delta^*(f \times g) = \Delta^*(A^*(f \boxtimes g)).$$

The student should check explicitly that this gives the formula stated above.

As we shall see, this rather indirect approach has some real advantages. For example, the naturality of each of the constituents is fairly clear, so the naturality of the cup product follows. That was already fairly from the explicit formula, but the more abstract formulation shows us how to proceed in other circumstances—as for example in cellular theory—where we might not have such an explicit formula. More to the point, it allows us to prove the graded commutative law as follows.

First, consider the effect of switching factors in the morphism

$$\boxtimes : \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) \to \mathrm{Hom}(S_*(X) \otimes S_*(X), K)$$

Define $T : S_*(X) \otimes S_*(X) \to S_*(X) \otimes S_*(X)$ by $T(\sigma \otimes \tau) = (-1)^{rs} \tau \otimes \sigma$ where σ, τ have degrees r, s respectively. This is a chain morphism since

$$\begin{aligned}\partial(T(\sigma \otimes \tau)) &= (-1)^{rs}\partial(\tau \otimes \sigma) = (-1)^{rs}\partial\tau \otimes \sigma + (-1)^{rs+s}\tau \otimes \partial\sigma \\ T(\partial\sigma \otimes \tau) &= T(\partial\sigma \otimes \tau) + (-1)^r T(\sigma \otimes \partial\tau) \\ &= (-1)^{(r-1)s}\tau \otimes \partial\sigma + (-1)^{r+r(s-1)}\partial\tau \otimes \sigma.\end{aligned}$$

However, $(r-1)s = rs - s \equiv rs + s \bmod 2$ and $r + r(s-1) = rs$, so the two expressions are equal. Note that just twisting the factors without introducing a sign would not result in a chain morphism. Note also, that in degree zero, T just introduces the identity in homology

$$H_0(S_*(X) \otimes S_*(X)) \to H_0(S_*(X) \otimes S_*(X)).$$

Indeed, using a trivial example of the Künneth theorem, this is just the homomorphism $\mathbf{Z} \otimes Z \to \mathbf{Z} \otimes Z \cong \mathbf{Z}$ defined by $a \otimes b \mapsto b \otimes a \cong ba = ab$.

Similarly, $\mathrm{Hom}(S_*(X), K)$ is a cochain complex (so also a chain complex), and we have the analogous twisting morphism $T : \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) \to \mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K)$ defined by $f \otimes g \mapsto (-1)^{rs} g \otimes f$. It is easy to check that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) \arrow[r, "\boxtimes"] \arrow[d] & \mathrm{Hom}(S_*(X) \otimes S_*(X), K) \arrow[d, "T^*"] \\
\mathrm{Hom}(S_*(X), K) \otimes_K \mathrm{Hom}(S_*(X), K) \arrow[r, "\boxtimes"] & \mathrm{Hom}(S_*(X) \otimes S_*(X), K)
\end{tikzcd}$$

commutes, where both vertical morphisms result from twisting.

To complete the proof of the graded commutative law, it suffices to show that the two cochain morphisms

$$\begin{aligned}\Delta^* \circ A^* \circ T^*, \Delta^* A^* : \mathrm{Hom}(S_*(X) \otimes S_*(X), K) &\to \mathrm{Hom}(S_*(X \times X), K) \\ &\to \mathrm{Hom}(S_*(X), K)\end{aligned}$$

are cochain homotopic. Let X and Y be spaces, and define $t : X \times Y \to Y \times X$ by $t(x, y) = (y, x)$. Consider the diagram

$$\begin{tikzcd}
S_*(X \times Y) \arrow[r, "A"] \arrow[d, "t_*"'] & S_*(X) \otimes S_*(Y) \arrow[d, "T"] \\
S_*(Y \times X) \arrow[r, "A"] & S_*(Y) \otimes S_*(X)
\end{tikzcd}$$

Each of the two routes from the upper left corner to the lower right corner is a natural chain map which induces the same isomorphism in homology in degree zero—the identity of $\mathbf{Z} \to \mathbf{Z}$. Hence, by the acyclic models theorem applied to the two functors $S_*(X \times Y)$ and $S_*(Y) \otimes S_*(X)$ on the category of pairs (X, Y) of spaces, it follows that the two routes yield chain homotopic morphisms. However, since

$t \circ \Delta = \Delta : X \to X \times X$, the same is true for the induced morphisms $S_*(X) \to S_*(X \times X)$, and combining this with the previous result gives us what we want.

4. Calculation of Cup Products

In this section we calculate some cohomology rings for important spaces.

Note first that if X is path connected, then $H^0(X; K)$ is naturally isomorphic to K, so we may view $1 \in K$ as an element of $H^0(X; K)$. It is not hard to see that it acts as the identity in the ring $H^*(X; K)$.

Spheres

First, note that $H^*(S^n; K)$ is easy to determine. Namely, $H^0(S^n; K) = K$, and since $H_k(S^n) = 0$ for $0 < k < n$ and $\mathbf{Z}$ for $k = n$, the universal coefficient theorem tells us that $H^k(S^n; K) = 0$ for $0 < k < n$ and $H^n(S^n; K) = K$. Let X be a generator of $H^n(S^n; K)$. By degree considerations, we must have $X \cup X = 0$. Hence, as a ring,

$$H^*(S^n; K) \cong K[X]/(X^2) \qquad \deg X = n.$$

Such a ring is called a *truncated polynomial ring.*

The 2-Torus

As above, since $H_*(T^2)$ is free, the univeral coefficient theorem yields

$$H^0(T^2; K) = K, H^1(T^2; K) = KX \oplus KY, H^2(T^2; K) = KZ$$

where we have chosen names, X, Y, Z for generators in the indicated degrees. (1 is the generator in degree 0.) Because of degree considerations

$$u \cup v = -v \cup u$$

for any elements of degree 1. In particular, if 2 is not a zero divisor in K, then

$$X \cup X = Y \cup Y = 0.$$

This will take care of $K = \mathbf{Z}$ or any field not of characteristic 2. We shall show that for the torus, these squares are zero in any case. We also have

$$X \cup Y = -Y \cup X$$

and we shall show that up to a sign these are the same as Z. The cohomology ring is an example of what is called an *exterior algebra* on the generators X, Y.

Next consider the diagram below representing T^2 as an identification space with the indicated *singular simplices.*

As usual, $H_1(T^2)$ is generated by the homology classes of the cycles e_1 and e_2. In addition, $f_1 - f_2$ is a cycle generating $H_2(T^2)$. There are several ways to see that. The simplest is to note that if we subdvide the torus further (as indicated in the above diagram), we may view it as a simplicial complex. In our previous calculation of the homology of the torus, we used exactly that simplicial decomposition. A true simplicial cycle generating $H_2(T^2)$ is obtained from $f_1 - f_2$ by sufficiently many subdivisions, and subdivision is chain homotopic to the identity.

Suppose g, h are two 1-cocycles. By the universal coefficient theorem, the cohomology classes $\overline{g}, \overline{h}$ are completely determined by $g(e_i), h(e_i)$ since in this case the Ext terms vanish and α is an isomorphism. (Remember that α amounts to evaluation at the leve of cocycles and cycles!) Similarly, the cohomology class of $g \cup h$ is completely determined by its values on $f_1 - f_2$. However,

$$\begin{aligned}(g \cup h)(f_1 - f_2) &= g(f_1 \circ [e_0, e_1])h(f_1 \circ [e_1, e_2]) - g(f_2 \circ [e_0, e_1])h(f_2 \circ [e_1, e_2) \\ &= g(e_1)h(e_2) - g(e_2)h(e_1).\end{aligned}$$

Suppose that g represents X and h represents Y, i.e., suppose $g(e_1) = 1, g(e_2) = 0, h(e_1) = 0, h(e_2) = 1$. Then

$$(g \cup h)(f_1 - f_2) = 1$$

and $g \cup h$ represents a generator of $H^2(T^2; K)$. On the other hand, suppose $g = h$ both represent X. Then the same calculations shows that $(g \cup g)(f_1 - f_2) = 0 - 0 = 0$. It follows that $X \cup X = 0$, and similarly $X \cup Y = 0$.

4.1. Cohomomology Rings of Products. Since $T^2 = S^1 \times S^1$, another approach to determining its cohomology ring is to study in general the cohomology ring of a product $X \times Y$.

We shall concentrate on the case $K = \mathbf{Z}$ *and* $H_*(X)$ and $H_*(Y)$ are free *or* K is a field. In either case, we have an isomorphism

$$H^*(X \times Y; K) \cong \mathrm{Hom}_K(H_*(X \times Y; K), K)$$

We also assume that the homology of each space is finitely generated in each degree. Under these assumptions,

$$H_*(X; K) \otimes_K H_*(Y; K) \xrightarrow{\times} H_*(X \times Y; K)$$

is an isomorphisms. For $\times$ is a monomomphism in either case by the universal coefficient theorem, and because the Tor terms vanish, it is an isomophism. Dualizing yields

$$\mathrm{Hom}_K(H_*(X; K) \otimes_K H_*(Y; K), K) \to \mathrm{Hom}_K(H_*(X \times Y; K), K) \cong H^*(X \times Y; K).$$

On the other hand it is also true that

$$\mathrm{Hom}_K(H_*(X;K),K) \otimes_K \mathrm{Hom}_K(H_*(Y;K),K) \xrightarrow{\boxtimes} \mathrm{Hom}_K(H_*(X;K) \otimes_K H_*(Y;K),K)$$

is an isomorphism. For, in each degree this amounts to explicit homomorphisms

$$\mathrm{Hom}_K(H_r,K) \otimes_K \mathrm{Hom}_K(H_s,K) \to \mathrm{Hom}_K(H_r \otimes_K H_s,K)$$

which may be shown by induction on the rank of H_r and H_s to be an isomorphism. If either H_r or H_s is of rank 1 over K, the result is obvious, and the inductive step follows using additivity of the functors.

Putting all of the above isomorphisms together, we get

$$H^*(X \times Y;K) \cong H^*(X;K) \otimes_K H^*(Y;K)$$

or more explicitly

$$H^n(X \times Y;K) \cong \bigoplus_{r+s=n} H^r(X;K) \otimes_k H^s(Y;K).$$

Hence, to describe the cup product on the left, it suffices to determine what it becomes on the right. The rule is quite simple.

$$(45) \qquad (u_1 \otimes v_1) \cup (u_2 \otimes v_2) = (-1)^{\deg v_1 \deg u_2} u_1 \cup u_2 \otimes v_1 \cup u_2.$$

In other words, mutiply generating tensor products in the obvious way by multiplying their factors but also introduce a sign. The sign is thought of as resulting from 'moving' the second factor on the left 'past' the first factor on the right.

We shall verify this rule below, but first note that it gives the same result as above for $T^2 = S^1 \times S^1$. For, let $H^1(S^1;K) = Kx$ so of necessity $x \cup x = 0$. Put

$$X = x \otimes 1, Y = 1 \otimes x \in H^1(S^1;K) \otimes_K H^0(S^1;K) \oplus H^0(S^1;K) \otimes_K H^1(S^1;K) \cong H^1(T^2;K).$$

Then

$$\begin{gathered}
X \cup Y = x \otimes 1 \cup 1 \otimes x = x \otimes x \\
Y \cup X = 1 \otimes x \cup x \otimes 1 = (-1)^{1 \cdot 1} x \otimes x = -X \cup Y \\
X \cup X = x \otimes 1 \cup x \otimes 1 = (x \cup x) \otimes 1 = 0 \\
Y \cup Y = 1 \otimes x \cup 1 \otimes x = 1 \otimes (x \cup x) = 0.
\end{gathered}$$

The proof of formula 45 follows from the following rather involved diagram which traces the morphisms at the level of singular chains needed to define the cup product. (You should also check that dualizing gives exactly what we want by means of the various isomorphisms described above. There are some added complications if $K \neq \mathbf{Z}$, so you should first work it out for ordinary singular theory.)

This diagram commutes up to chain homotopy by the acyclic models theorem applied to the category of pairs (X, Y). As we saw previously, with the specified models, the functor $S_*(X \times Y)$ is free. Also, as before, the Künneth Theorem shows that the functor

$$S_*(X) \otimes S_*(Y) \otimes S_*(X) \otimes S_*(Y)$$

is acyclic. It is easy to check that both routes between the ends induce the same morphism in homology in degree zero.

CHAPTER 11

Manifolds and Poincaré duality

1. Manifolds

The homology $H_*(M)$ of a manifold M often exhibits an interesting symmetry. Here are some examples.

$$M = S^1 \times S^1 \times S^1 : \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}, H_2 = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}, H_3 = \mathbf{Z}$$
$$M = S^2 \times S^3 : \quad H_0 = \mathbf{Z}, H_1 = 0, H_2 = \mathbf{Z}, H_3 = \mathbf{Z}, H_4 = 0, H_5 = \mathbf{Z}$$
$$M = \mathbf{R}P^3 : \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z}/2\mathbf{Z}, H_2 = \mathbf{Z}$$

Note that the symmetry $H_i \cong H_{n-1}$ is complete in the first two cases, and is complete in the third case if we just count ranks. Also, $H_n(M) = \mathbf{Z}$ if n is the dimension of the manifold in these cases. It turns out that all these manifolds are orientable in a sense to be made precise below. (In fact orientability is tantamount to $H_n(M) = \mathbf{Z}$, and as we shall say an orientation may be thought of as a choice of generator for this group.) The example of the Klein bottle illustrates what can happen for a non-orientable manifold.

$$M = K : \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, H_2 = 0.$$

(You should work that out for yourself.) However, the universal coefficient theorem shows that

$$H_0(K; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}, H_1(K; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, H_2(K; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$$

which has the same kind of symmetry. Such a manifold would be called $\mathbf{Z}/2\mathbf{Z}$-orientable. Finally, a space which is not a manifold will not generally exhibit such symmetries.

$$X = S^2 \vee S^3 : \quad H_0 = \mathbf{Z}, H_1 = \mathbf{Z}, H_2 = \mathbf{Z}, H_3 = \mathbf{Z}.$$

1.1. Orientability of Manifolds. The naive notion of orientability is easy to understand in the context of finite simplicial complexes. (The manifold in this case would be compact.) Roughly, we assume each n-simplex of the manifold is given an order which specifies an orientation for that simplex. Moreover, we assume the orientations of the simplices can be chosen coherently so that induced orientations on common faces cancel. In this case, the sum of all the n-simplices will be a cycle which represents a homology class generating $H_n(M)$. To define orientability in the context of singular homology is more involved. We shall do this relative to a coefficient ring R so we may encompass things like the example of the Klein bottle with $R = \mathbf{Z}/2\mathbf{Z}$. (In that case, if we used a simplicial decomposition, the sum of all the simplices would be a cycle modulo 2 because when the same face showed up twice in the boundary, cancellation would occur because $1 + 1 = 0$ in $\mathbf{Z}/2\mathbf{Z}$.)

In all that follows let M be a *not necessarily connected n-manifold.* If $x \in M$, abbreviate $M - x = M - \{x\}$.

LEMMA 11.1. *Let $x \in M$.*

$$H_i(M, M - x; R) = \begin{cases} R & \textit{if } i = n, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Choose an open neighborhood U of x which is homeomorphic to an n-ball. $M - U$ is closed and contained in $M - x$ which is open. Hence, excising $M - U$ yields

$$H_i(U, U - x; R) \cong H_(M, M - x).$$

However, since $U - x$ has S^{n-1} as a retract, the long exact sequence for reduced homology shows that

$$H_i(M, M - x) \cong \tilde{H}_{i-1}(S^{n-1}; R) = R \ (i = n) \quad \text{or 0 otherwise.}$$

(Why do we still get a long exact sequence for relative homology with coefficients? Check this for yourself.) □

A local R-orientation at a point x in an n-manifold M is a choice μ_x of generator for $H_n(M, M - x; R)$ (as R-module). For $R = \mathbf{Z}$ there would be two possible choices, for $\mathbf{R} = \mathbf{Z}/2\mathbf{Z}$ only one possible choice, and in general, the number of possible choices would depend on the number of units in the ring R. You should think of μ_x as a generic choice for orientations of spheres in a euclidian neighborhood of x centered at x. Such a sphere may be considered (up to homeomorphism) to be the boundary of a simplex, and specifying an orientation for the simplex (as an ordering of its vertices) is the same thing as specifying a generator for the homology of its boundary.

Introduce the following notation. If $L \subset K \subset M$, let $\rho_{K,L}$ denote the functorial homomoprhism $H_*(M, M-K; R) \to H_*(M, M-L; R)$. If K is fixed, just use ρ_L. Call this homomorphism 'restriction'.

We shall say a that a choice of local R-orientations for each point $x \in M$ is continuous if for each point of M, there is a neighborhood N of x and an element $\mu_N \in H_n(M, M-N; R)$ such that $\rho_y(\mu_N) = \mu_y$ for each $y \in N$. We shall say that M is R-orientable if there is a continuous choice of local R-orientations for each point.

Example 11.2. Let $M = S^n$ and choose a generator $\mu \in H_n(S^n; R)$. Let $\mu_x = \rho_x(\mu)$ where

$$\rho_x = \rho_{S^n,x} : H_n(S^n; R) = H_n(S^n, S^n - S^n; R) \to H_n(S^n, S^n - x; R).$$

Take $N = S^n$ for every point. Note that the long exact sequence for the pair $(S^n, \{x\})$ shows that ρ_x is an isomorphism. Hence, μ_x is in fact a generator of $H_n(S^n, S^n - x; R)$ for every $x \in S^n$.

It is not hard to see that if M is R-orientable, then any open subset U of M inherits an R-orientation. Hence, $\mathbf{R}^n$ which is an open subset of S^n is also R-orientable.

Since $H_n(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$, it is not too hard to see that $\mathbf{R}P^n$ is $\mathbf{Z}/2\mathbf{Z}$-orientable.

Our aim is to show—at least in the case of a compact R-orientable manifold—that $H_n(M; R) = R\mu$ for some μ and $\rho_x(\mu) = \mu_x$ for each $x \in M$. If M is not compact, we show instead that this is true for $H_n(M, M-K; R)$ for every compact subset K of M. (If M is compact, this would include $K = M, M-K = \emptyset$.) First we need a technical lemma.

Lemma 11.3. *Let M be an n-manifold and K a compact subspace.*

- $H_i(M, M-K; R) = 0$ *for* $i > n$.
- $\alpha \in H_n(M, M-K; R)$ *is zero if and only if* $\rho_x(\alpha) = 0$ *for each* $x \in K$.

We shall abbreviate the second statement by saying that the elements of K *detect* the degree n homology of $(M, M-K)$.

Proof. **Step 1**. Let $M = \mathbf{R}^n$ and let K be a compact convex subspace.

Let $x \in K$. Enclose K in a large closed ball B centered at x. Then there is a retraction of $R^n - x$ onto $S = \partial B$ and its restriction to $R^n - K$ and $R^n - B$ are also retractions. It follows that the restrictions in

$$H_i(R^n, S) \cong H_i(R^n, R^n - B; R) \cong \\ H_i(R^n, R^n - K; R) \cong H_i(R^n, R^n - x; R)$$

are all isomorphisms, which proves the second statement. The isomorphism

$$H_i(R^n, S; R) \cong H_{i-1}(S; R)$$

proves the first statement.

Step 2. Suppose that M is arbitrary, and the lemma has been proved for compact subsets K_1, K_2 and $K_1 \cap K_2$. Let $K = K_1 \cup K_2$. Then $M - K = (M - K_1) \cap (M - K_2)$, $M - K_1 \cap K_2 = (M - K_1) \cup (M - K_2)$, and there is a *relative* Mayer–Vietoris sequence

$$\to H_i(M, M - K; R) \to H_i(M, M - K_1; R) \oplus H_i(M, M - K_2; R) \to \\ H_i(M, M - K_1 \cap K_2; R) \to H_{i-1}(M, M - K; R) \to \dots$$

It is easy to establish the first statement for K from this. Also,

$$H_n(M, M - K; R) \to H_n(M, M - K_1; R) \oplus H_n(M, M - K_2; R)$$

is a monomorphism. Suppose $\rho_{K,x}(\alpha) = 0$ for each $x \in K = K_1 \cup K_2$. It follows that $\rho_{K,K_1}(\alpha) = 0$ and similarly for K_2. Hence, $\alpha = 0$.

Step 3. $M = R^n$ and K is a finite union of convex compact subspaces. Use steps 1 and 2.

Step 4. Let $M = \mathbf{R}^n$ and suppose K is any compact subspace. To prove the result in this case, let $\alpha \in H_i(M, M - K; R)$. Let $a \in S_i(M)$ be a chain with boundary $\partial a \in S_i(M - K)$ which represents α. Choose a covering of K by closed balls $B_i, i = 1, r$ which are disjoint from $|\partial a|$. (This is possible since both sets are compact.)

Then $\partial a \in S_i(M - B)$ where $B = B_1 \cup \dots \cup B_r$, so a represents and element $\alpha' \in H_i(M, M - B; R)$ which restricts to $\alpha \in H_i(M, M - K; R)$. If $i > n$, then by Step 3, $\alpha' = 0$, so $\alpha = 0$. Since α could have been anything, this shows $H_i(M, M - K; R) = 0$ for $i > n$. Suppose $i = n$. Suppose $\rho_{K,x}(\alpha) = 0$ for each $x \in K$. Then $\rho_{B,x}(\alpha') = 0$ for each $x \in K$. If we can show that the same is true for each $x \in B$, then it will follow from Step 3 that $\alpha' = 0$ and so $\alpha = 0$. To see this, first note that we may assume that each B_i intersects K non-trivially or else we could have left it out. If $x \in B_i$ then there is a $y \in K \cap B_i$ and we have isomorphisms

$$H_n(M, M - y; R) \leftarrow H_n(M, M - B_i) \rightarrow H_n(M, M - x; R).$$

It follows that if $\rho_{B,y}(\alpha') = \rho_{B,B_i}(\rho_{B_i,y}(\alpha') = 0$, then $\rho_{B,x}(\alpha') = 0$.

Step 5. Suppose M is arbitrary and K is contained in an open euclidean neighborhood U homeomorphic to $\mathbf{R}^n$. Excising $M - U$ (which is contained in the interior of $M - K$) we get

$$H_i(U, U - K; R) \cong H_i(M, M - K; R).$$

We can now apply Step 4.

Step 6. The general case. We may assume $K = K_1 \cup K_2 \cup \cdots \cup K_r$ where each K_i is as in Step 5. The same would be true of each intersection of K_i with the union of those that preceded it. Hence, we may apply Step 2 and Step 5.

□

THEOREM 11.4. *Let M be an R-orientable n-manifold with local orientations $\mu_x, x \in M$. Let K be any compact subspace. There exists a unique element*

$$\mu_K \in H_n(M, M - K; R) \quad \textit{such that } \rho_x(\mu_K) = \mu_x$$

for each $x \in K$. In particular, if M is compact itself, there is a unique element $\mu_M \in H_n(M; R)$ such that $\rho_x(\mu_M) = \mu_x$ for each $x \in M$.

In case M is compact, μ_M is called the fundamental class of M. As we shall see later, if M is also connected, then μ_M is a generator of $H_n(M; R)$ as an R-module. If M is not connected, then $H_n(M; R)$ will turn out to be a direct sum of copies of R, one for each component of M. μ_M will be a sum of generators of this direct sum, one for each component. This will correspond to choosing a collection of orientations in the usual sense for the components of M.

PROOF. By the lemma above, such an element μ_K is unique. To show existence, argue as follows.

Step 1. According to the definition of continuity of a choice of local orientations, there is a neighborhood N with the right property. If K is contained in such an N, restricting μ_N to an element $\mu_K \in H_n(M, M - K; R)$ will work.

Step 2. Suppose $K = K_1 \cup K_2$, both compact, and the theorem has been verified for K_1 and K_2. From the lemma, the relative Mayer–Vietoris sequence yields the exact sequence

$$0 \to H_n(M, M - K; R) \to H_n(M, M - K_1; R) \oplus H_n(M, M - K_1; R) \\ \to H_n(M, M - K_1 \cap K_2; R) \to 0.$$

As usual, the homomorphism on the right maps (μ_{K_1}, μ_{K_2}) to

$$\rho_{K_1, K_1 \cap K_2}(\mu_{K_1} - \rho K_2, K_1 \cap K_2(\mu_{K_2}) \in H_n(M, M - K_1 \cap K_2; R).$$

Further restricting these elements to $H_n(M, M - x; R)$ for $x \in K_1 \cap K_2$ yields zero, so by the lemma, the above difference is zero. Hence, by the exactness of the sequence, there is a unique $\mu_K \in H_n(M, M - K; R)$ such that $\rho_{K,K_1}(\mu_K) = \mu_{K_1}$ and $\rho_{K,K_2}(\mu_K) = \mu_{K_2}$, and it is easy to check that μ_K has the right property.

Step 3 Let K be an arbitrary compact subspace. By the continuity condition, we may cover K by neighborhoods N with appropriate elements μ_N. By taking smaller neighborhoods if necessary, we may assume that each N is compact. Since the sets $\overset{\circ}{\to} N$ cover K, we may pick out finitely many $N_i, i = 1 \dots, r$ which cover K. Let $K_i = N_i \cap K$, and now apply steps (i) and (ii) and induction. $\square$

2. Poincaré Duality

Assume in what follows that R is a ring with reasonable properties as described above, e.g., $R = \mathbf{Z}$ or R is a field.

The Poincaré Duality Theorem asserts that if M is a compact R-oriented n-manifold, then

$$H^r(M; R) \cong H_{n-r}(M; R).$$

(Then, we may use the universal coefficient theorem to relate the homology in degrees r and $r - 1$ to the homology in degree $n - r$. Think about it!) In order to describe the isomorphism we need another kind of product which relates cohomology and homology.

2.1. Cap Products.

First we need to establish some notation. As usual, let R be a commutative ring (usually $R = \mathbf{Z}$ or R is a field), and let $S_*(X; R) = S_*(X) \otimes R$ as usual. This is a chain complex which is free over R, and we have a natural isomorphsim

$$\mathrm{Hom}_R(S_*(X; R), R) \cong \mathrm{Hom}(S_*(X), R) = S^*(X; R)$$

provided by $f \in \mathrm{Hom}_R(S_*(X) \otimes R, R) \mapsto f' \in \mathrm{Hom}(S_*(X), R)$ where $f'(\sigma) = f(\sigma \otimes 1)$. Moreover, $c \otimes f \mapsto f(c)$ defines a homomorphism

(called a pairing)

$$S_q(X;R) \otimes_R S^q(X;R) \to R$$

for each q. Note the order of the factors on the left. At this point the order is arbitrary, but it will be important in what follows. (Compare this with the definition of the morphism $H^n(X;R) \to \mathrm{Hom}(H_n(X), R)$.) Define an extension of this pairing as follows. Let $q \le n$. For $c \in S_n(X;R), f \in H^q(X;R)$ define $c \cap f \in H_{n-q}(X;R)$ by

$$\sigma \cap f = f(\sigma \circ [e_0, \dots, e_q])\sigma \circ [e_q, \dots, e_n]$$

for σ a singular n-simplex in X and extending by linearity. (There is a slight abuse of notation here. We are identifying σ with the element $\sigma \otimes 1 \in S_n(X;R) = S_n(X) \otimes R$. These elements form an R-basis for $S_n(X;R)$.) Note that if $q = n$, this is just the evaluation pairing described above. This cap product may also be described abstractly but this requires some fiddling with signs. See the Exercises.

PROPOSITION 11.5. *The cap product satisfies the following rules.*

(1) $c \cap (f \cup g) = (c \cap f) \cap g$. *Also* $c \cap e = c$ *where* $e \in S^0(X;R)$ *is defined by* $e(\sigma) = 1$ *for each singular 0-simplex* σ.
(2) $(-1)^q \partial(c \cap f) = (\partial c) \cap f - c \cap \delta f$.
(3) *Let* $j : X \to X'$ *be a map,* $c \in S_*(X;R), f' \in S^*(X';R)$. *Then* $j_*(c \cap j^*(f')) = j_*(c) \cap f'$.

Notes: The first statement may be interpreted as saying that $S_*(X;R)$ is a right module over the ring $S^*(X;R)$. To make sense of this, we let $c \cap f = 0$ if $q = \deg f > n = \deg c$. Note that e is the identity element of $S^*(X;R)$. The distributive laws for $c \cap f$—required for a module—are automatic since it is a pairing. A consequence of the second statement is that if f is a cocycle and c is a cycle, then $c \cap f$ is also a cycle and its homology class depends only on the classes of c and f respectively. Hence, it defines a pairing between cohomology and homology

$$H_n(X;R) \otimes H^q(X;R) \to H_{n-q}(X;R)$$

denoted $\bar{c} \cap \overline{f} = f(c)$. This pairing satisfies the rules asserted in the first statement.

PROOF. The statement about e is immediate from the formula.

To prove the associativity assertion, proceed as follows. Let σ be a singular n-simplex. Let $q_1 = \deg f, q_2 = \deg g$, let $q = q_1 + q + 2$, and let $r = n - q$. By linearity, it suffices to prove the formula if $c = \sigma$ is a singular n-simplex.

$$\begin{gathered}\sigma \cap (f \cup g)\\ = (f \cup g)(\sigma \circ [e_0, \dots, e_q])\sigma \circ [e_q, \dots, e_n]\\ = f(\sigma \circ [e_0, \dots, e_q] \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_0, \dots, e_q] \circ [e_{q_1}, \dots, e_q])\sigma \circ [e_q, \dots, e_n]\\ = f(\sigma \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_{q_1}, \dots, e_q])\sigma \circ [e_q, \dots, e_n].\end{gathered}$$

On the other hand,

$$\begin{gathered}\sigma \cap f\\ = f(\sigma \circ [e_0, \dots, e_{q_1}])\sigma \circ [e_{q_1}, \dots, e_n]\\ (\sigma \cap f) \cap g\\ = f(\sigma \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_{q_1}, \dots, e_n] \circ [e_0, \dots, e_{q_2}])\sigma \circ [e_{q_1}, \dots, e_n] \circ [e_{q_2}, \dots, e_q]\\ = f(\sigma \circ [e_0, \dots, e_{q_1}])g(\sigma \circ [e_{q_1}, \dots, e_q]])\sigma \circ [e_q, \dots, e_n].\end{gathered}$$

To prove the boundary formula, argue as follows. First note that any r-chain $c' \in S_r(X; R)$ is completely determined if we know $h(c')$ for every q cycle $h \in S^r(X; R)$. Now

$$\begin{aligned} h(\partial(c \cap f)) &= (\delta h)(c \cap f) = (c \cap f) \cap \delta h\\ &= c \cap (f \cup \delta h) = c \cap ((-1)^q(\delta(f \cup h) - \delta f \cup h))\\ &= (-1)^q(\delta(f \cup h)(c) - c \cap (\delta f \cup h))\\ &= (-1)^q((f \cup h)(\partial c) - (c \cap \delta f) \cap h)\\ &= (-1)^1((\partial c) \cap f - c \cap \delta f) \cap h\\ &= h((-1)^q(\partial c \cap f - c \cap \delta f))\end{aligned}$$

as required.

Note: If you use a more abstract definition of the cap product, then one introduces a suitable sign in the cap product. With that sign, the proof comes down to the assertion that the defining morphism is just a chain homomorphism.

The last formula is left as an exercise for the student. □

2.2. Poincaré Duality.

THEOREM 11.6. *Let M be a compact oriented R-manifold with fundamental class $\mu_M \in H_n(M; R)$. Then the homomorphism*

$$H^q(M; R) \to H_{n-q}(M; R)$$

defined by $a \mapsto \mu_M \cap a$ is an isomorphism.

Note that if $R = \mathbf{Z}$, then by the universal coefficient theorem for cohomology, we get a (non-natural) isomorphism

$$\mathrm{Hom}(H_q(M), \mathbf{Z}) \oplus \mathrm{Ext}(H_{q-1}(M), \mathbf{Z}) \cong H_{n-q}(M)$$

In particular, if $H_*(M)$ is free, then

$$H_q(M) \cong \mathrm{Hom}(H_q(M), Z) \cong H_{n-1}(M)$$

as we observed previously. If R is a field, then

$$H_q(M; R) \cong \mathrm{Hom}_R(H_q(M; R), R) \cong H_{n-1}(M; R).$$

We will prove the Poincaré Duality Theorem later, but first we give some important applications.

3. Applications of Poincaré Duality

3.1. Cohomology Rings of Projective Spaces.

THEOREM 11.7. (1) *$H^*(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}[X]/(X^{n+1})$ where $\deg X = 2$.*

(2) *$H^*(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}[X]/(X^{n+1})$ where $\deg X = 1$.*

(3) *$H^*(\mathbf{H}P^n; \mathbf{Z}) \cong \mathbf{Z}[X]/(X^{n+1})$ where $\deg X = 4$.*

PROOF. We shall do the case of $\mathbf{C}P^n$.

We know the result for $n = 1$ since $\mathbf{C}P^1 \simeq S^2$. Assume the result is true for $1, \dots, n-1$. In our calculation of $H_*(\mathbf{C}P^n)$, we in effect showed that $H_k(CP^{n-1}) \to H_k(\mathbf{C}P^n)$ is an isomorphism for $0 \le k \le 2n-2$. (This is also immediate from the calculation of the homology using the cellular decomposition.) Since $H_k(\mathbf{C}P^m)$ is trivial is odd degrees, the universal coefficient theorem for cohomology shows us that the induced homomorphism

$$H^{2k}(\mathbf{C}P^n; \mathbf{Z}) = \mathrm{Hom}(H_{2k}(\mathbf{C}P^n); \mathbf{Z}) \xrightarrow{\rho^{2k}} H^{2k}(\mathbf{C}P^{n-1}; \mathbf{Z}) = \mathrm{Hom}(H_{2k}(\mathbf{C}P^{n-1}); \mathbf{Z})$$

is an isomorphism. Choose X a generator of $H^2(\mathbf{C}P^n; \mathbf{Z})$. $X' = \rho^2(X)$ generates $H^2(\mathbf{C}P^{n-1}; \mathbf{Z})$, so by induction, $(X')^{n-1}$ generates $H^{2n-2}(\mathbf{C}P^{n-1}; \mathbf{Z})$. However, $X'^{n-1} = \rho(X)^{n-1} = \rho(X^{n-1}$, so X^{n-1} generates $H^{2n-2}(\mathbf{C}P^n; \mathbf{Z})$. Now apply the Poincaré Duality isomorphism

$$a \in H^{n-1}(\mathbf{C}P^n; \mathbf{Z}) \mapsto \mu \cap a \in H_2(\mathbf{C}P^n).$$

It follows that $\mu \cap X^{n-1}$ generates $H_2(X)$. Hence, the evaluation morphism must yield

$$\mu \cap X^{n-1} \mapsto (\mu \cap X^{n-1}) \cap X = \pm 1$$

Hence,

$$\mu \cap X^n = \mu \cap (X^{n-1} \cup X) = \pm 1$$

so X^n generates $H^{2n}(\mathbf{C}P^n; \mathbf{Z})$.

The argument for $\mathbf{H}P^n$ is essentially the same. The argument for $\mathbf{R}P^n$ is basically the same except that homology and cohomology have coefficients in $\mathbf{Z}/2\mathbf{Z}$. You should write out the argument in that case to make sure you understand it. □

You may recall the following result proved in a special case by covering space theory.

THEOREM 11.8. *preserving map* $f : S^n \to S^m$ *with* $0 \le m < n$.

The case we did earlier was $m = 1, n > 1$. The argument was that if we have such a map $f : S^n \to S_1$, it would induce a map $\overline{f} : \mathbf{R}P^n \to \mathbf{R}P^1 = S^1$ which would by the antipode property take a nontrivial loop in $\mathbf{R}P^n$ into a nontrivial loop in $\mathbf{R}P^1$. This would contradict what we know about the fundamental groups of those spaces.

The Borsuk–Ulam Theorem had many interesting consequences which you should now review along with the theorem.

PROOF. Assume $1 < m < n$. As in the previous case, we get an induced map and a diagram

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & S^m \arrow[d] \\ \mathbf{R}P^n \arrow[r, "\overline{f}"] & \mathbf{R}P^m \end{tikzcd}$$

By covering space theory, the fundamental groups of both projective spaces are $\mathbf{Z}/2\mathbf{Z}$ and by the antipode property, the induced map of fundamental groups is an isomorphism. Since $H_1(X) \cong \pi_1/[\pi_1, \pi_1]$, it follows that the first homology group of both projective spaces is $\mathbf{Z}/2\mathbf{Z}$ and the induced morphism is an isomorphism. Thus, the universal coefficient theorem for cohomology allows us to conclude that

$$\overline{f}^* : H^1(\mathbf{R}P^m; \mathbf{Z}/2\mathbf{Z}) \to H^1(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z})$$

is an isomorphism, so a generator X_m of the former goes to a generator X_n of the latter. Hence,

$$0 = \overline{f}^*({X_m}^n) = (\overline{f}^*(X_m))^n = (X_n)^n$$

which contradicts our calculation of $H^*(\mathbf{R}P^n; \mathbf{Z}/2\mathbf{Z})$.

□

As mentioned previously, there is a non-associative division algebra of dimension 8 called the Cayley numbers. Also, n-dimensional projective space ofer the Cayley numbers may be defined in the usual way. It is a compact $8n$-dimensional manifold. The above argument shows that its cohomology ring is a truncated polynomial ring on a generator of degree 8.

3.2. Aside on the Hopf Invariant. Earlier in this course, we discussed the homotopy groups $\pi_n(X, x_0)$ of a space X with a base point x_0. We mentioned that these are abelian for $n > 1$, and that $\pi_m = 0$ for $X = X^n$ and $0 < m < n$. In this section we shall show that $\pi_m(S^n) \neq 0$ for $m = 2n - 1$. The cases $n = 2, 4, 8$ use the above calculations of cohomology rings for projective spaces, so that is why we include this material here.

Consider a map $f : S^{2n-1} \to S^n$. Since $S^{2n-1} = \partial D^{2n}$, we may form the adjunction space $D^{2n} \sqcup_f S^n$. We showed earlier that

$$\begin{aligned} H_{2n}(D^{2n} \sqcup_f S^n) &= \mathbf{Z} \\ H_n(D^{2n} \sqcup_f S^n) &= H_n(S^n) = \mathbf{Z} \\ H_0(D^{2n} \sqcup_f S^n) &= \mathbf{Z} \\ H_k(D^{2n} \sqcup_f S^n) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Hence, the universal coefficient theorem for cohomology shows that

$$\begin{aligned} H^{2n}(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= \mathbf{Z} \\ H^n(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= \mathbf{Z} \\ H^0(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= \mathbf{Z} \\ H^k(D^{2n} \sqcup_f S^n; \mathbf{Z}) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Let a_k be a generator of $H^k(D^{2n} \sqcup_f S^n; \mathbf{Z})$ for $k = n, 2n$. Then $a_n \cup a_n = H(f)a_{2n} \in H^{2n}(D^{2n} \sqcup_f S^n; \mathbf{Z})$ where $H(f)$ is an integer called the *Hopf invariant* of the map f. Note that $H(f)$ depends on the generators a_n, a_{2n}, but that choosing different generators will at worst change its sign. Since all the constructions we have made are invariant under homotopies, $H(f)$ is an invariant of the homotopy class of the map $f : S^{2n-1} \to S^n$, i.e., of the element of $\pi_{2n-1}(S^n)$ that it defines.

Consider the special case $n = 2$ and $f : S^3 \to S^2$ is the attaching map for the identification $\mathbf{C}P^2 = D^4 \sqcup_f \mathbf{C}P^1$. ($f$ is the quotient map where $S^2 = \mathbf{C}P^1$ is identified as the orbit space of the action of S^1 on S^3 discussed previously.) Since $X \cup X$ generates $H^4(\mathbf{C}P^2; \mathbf{Z})$ the Hopf invariant of the attaching map is ± 1.

We have not discussed the group operation in π_m for $m > 1$, but it is possible to show that $f \to H(f)$ defines a group homomorphism $\pi_{2n-1}(S^n) \to \mathbf{Z}$. Hence, the above argument shows that this homomorphism is onto. Hence, $\pi_3(S^2)$ has a direct summand isomorphic to $\mathbf{Z}$.

A similar argument works for quaternionic projective space. In that case $\mathbf{H}P^1 = D^4 \sqcup_g \mathbf{H}P^0$ which is a 4-cell adjoined to a point, i.e., $\mathbf{H}P^1 \simeq S^4$. Hence, the attaching map f in $\mathbf{H}P^2 = D^8 \sqcup_f \mathbf{H}P^1$ has hopf invariant ± 1.

In fact the above argument will work for any finite dimensional real division algebra (associative or not). In particular, it also works for the Cayley numbers, so there is a map $f : S^{15} \to S^8$ of Hopf invariant 1. Moreover, one can use the theory of the Hopf invariant to help show that there are no other finite dimensional real division algebras.

For example, it is not hard to see that if n is odd, any map $f : S^{2n-1} \to S^n$ has $H(f) = 0$. This follows immediately from the fact that in the odd case $a_n \cup a_n = -a_n \cup a_n = 0$.

If n is even, there is always a map of Hopf invariant 2. For consider the identification

$$S^n \times S^n = D^{2n} \sqcup_g (S^n \vee S^n).$$

Define the 'folding map' $S^n \vee S^n \to S^n$ by sending each component identically to itself. Let $f : S^{2n-1} \to S^n$ be the composition of $g : S^{2n-1} \to S^n \vee S^n$ with the folding map. Let X be the quotient space of $D^{2n} \sqcup_g S^n \vee S^n$ under the relation which identifies points of $S^n \vee S^n$ which fold to the same point. Then we get a commutative diagram

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "g"] \arrow[d, "="'] & D^{2n} \sqcup_g S^n \vee S^n = S^n \times S^n \arrow[d, "\phi"] \\
S^{2n-1} \arrow[r, "f"] & X
\end{tikzcd}$$

where ϕ is the quotient map (induced by folding). It is not hard to see that this presents X as $D^{2n} \sqcup_f S^n$, and $\phi^* : H^{2n}(D^{2n} \sqcup S^n; \mathbf{Z}) \to H^{2n}(S^n \times S^n; \mathbf{Z})$ is an isomorphism. Also, if c_n generates $H^n(D^{2n}(\sqcup_f S^n) = H^n(S^n; \mathbf{Z})$, then $\phi^*(c_n) = a_n + b_n$ for appropriate generators of $H^n(S^n \times S^n; \mathbf{Z}) = H^n(S^n \vee S^n; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. Since in $H^*(S^n \times S^n; \mathbf{Z})$, we have ${a_n}^2 = {b_n}^2 = 0$, we have $(a_n + b_n)^2 = 2a_n b_n$. (Since n is even, a_n, b_n commute.) But $a_n b_n$ generates $H^{2n}(S^n \times S^n; \mathbf{Z})$ so it follows that $c_n \cup c_n$ is twice a generator. Hence, its Hopf invariant is ± 2.

Note that the above argument shows that $\pi_{2n-1}(S^n) \to \mathbf{Z}$ at worst goes onto the subgroup of even integers if n is even. Since this is also isomorphic to $\mathbf{Z}$, we see that $\mathbf{Z}$ is a direct summand of that homotopy group for n even. Using cohomology operations called the Steenrod squares, which are a generalization of the cup product, one can show that a nessary condition for the homomorphism $\pi_{2n-1}(S^n) \to \mathbf{Z}$ to be onto, i.e. for there to be an element of Hopf invariant 1, is for n to be a power of 2. In 1960, J. F. Adams settled the question by showing that the only cases in which there exist elements of Hopf invariant 1 are the ones described above, i.e. when $n = 2, 4, or\, 8$, but we leave that fact for you to explore in another course or by independent reading.

4. Cohomology with Compact Supports

In order to prove the Poincaré Duality theorem for compact manifolds, we shall need to use an argument which reduces it to constituents of a covering by open Euclidean neighborhoods. Such sets, unfortunately, are not compact manifolds, so we must extend the theorem beyond the realm of compact manifolds in order to prove it. However, in the non-compact case, we saw that we had to deal with fundamental classes $\mu_K \in H_n(M, M - K; R)$ with K compact rather than one fundamental class $\mu_M \in H_n(M; R)$. The basic result, as we stated it earlier, is not true for an individual K, but as we shall see, it is true 'in the limit'.

4.1. Direct Limits. Let I be a partially ordered set. The example which shall be most important for us is the set of compact subsets of a space X ordered by inclusion. A partially ordered set may be considered a category with the objects the elements of the set and the morphisms the pairs (j, i) such that $j \geq i$ in the ordering. We consider i to be the source of the morphism and j its target. Thus, for each pair j, i of elements in I, either $\mathrm{Hom}(j, i)$ is empty or $\mathrm{Hom}(j, i)$ consists of the single element (j, i). If $k \geq j, j \geq i$, we let $(k, j) \circ (j, i) = (k, i)$. It is easy to check that this composition satisfies the requirements for a category.

Let R be a commutative ring and let F be a functor from I to the category of R-modules. Thus, we are given an R-module $F(i)$ for each $i \in I$ and a module homomorphism $\phi_{i,j} : F(i) \to F(j)$ for each $j \geq i$ such that

$$\phi_{i,i} = \mathrm{Id} : F(i) \to F(i)$$

$$\phi_{k,j} \circ \phi_{j,i} \qquad \text{for } k \geq j \geq i.$$

A direct limit of such a functor consists of an R module M and module homomorphisms $\psi_i : F(i) \to M$ such that for $j \geq i$, $\psi_i = \psi_j \circ \phi_{j,i}$

and M and the collection of ϕ_i is universal for this property, i.e., given another R-module M' and module homomorphisms $\psi_i' : F(i) \to M'$ such that for $j \geq i$, $\psi_i' = \psi_j' \circ \phi_{j,i}$ then there is a unique module homomorphism $\rho : M \to M'$ such that $\psi_i' = \rho \circ \psi_i$ for each i in I.

The defintion of direct limit may be extended to an arbitrary functor from one category to another. We leave it to the student to fill in the details of the definition.

A direct limit of a functor F is unique up to unique isomorphism. This follows immediately from the universal mapping property by some diagram chasing. We leave it for the student to verify as an exercise. We use the notation $\varinjlim F$ to denote 'the' direct limit. A direct limit always exists. To see this consider first the case where the ordering of I is empty, i.e., $i \geq i$ for $i \in I$ are the only orderings. In this case, the direct limit is just the direct sum $\oplus_{i \in I} F(i)$ and $\eta_i : F(i) \to \bigoplus$ be the inclusion of the summand $FIi)$ in the direct sum. This collection of homomorphisms presents $\bigoplus$ as a direct limit. Assume now that I is any partially ordered set. In $\oplus_i F(i)$, consider the subgroup T generated by all elements of the form $\eta_j(\phi_{j,i}(x_i)) - \eta_i(x_i)$ for $x_i \in F(i)$ and $j \geq i$. Let $M = (\oplus_i F(i))/T$ and let $\psi_i : F(i) \to M$ be the homomorphism which composes the factor morphisms with η_i. We leave it to the student to check that M and the morphisms ψ_i yield a direct limit.

Let I be a partially ordered set. We say that I is *directed* if each pair $i, j \in I$ has an upper bound.

We say a sequence $F' \to F \to F''$ of functors on a partially ordered set I is exact if $F'(i) \to F(i) \to F''(i)$ is exact for each $i \in I$.

PROPOSITION 11.9. *Let I be a directed partially ordered set. If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of functors on I,* *then*

$$0 \to \lim_{\to} F' \to \lim_{\to} F \to \lim_{\to} F'' \to 0$$

is exact.

PROOF. We let the student verify that the sequence of limits is right exact. That does not even require that the set I be directed.

To show that it is exact on the left, i.e., that

$$\lim_{\to} F' \to \lim_{\to} F$$

is a monomorphism, we must use the directed property. Let

$$M' = \lim_{\to} F', \qquad M = \lim_{\to} F.$$

First note the following facts which hold in the directed case.

1. Any element $x \in M$ is of the form $\psi_i(x_i)$ for some $x_i \in F(i)$.

For, by the above construction, any element in M is of the form $x = \sum_j \psi_j(x_j)$ where all but a finite number of terms in the sum are zero. By the directed property we can choose an $i \geq j$ for all such j. Since $\psi_i(\phi_{i,j}(x_j)) = \phi_j(x_j)$ in M, it follows that

$$x = \psi_i(\sum_i \phi_{i,j}(x_j))$$

as claimed.

2. If $\psi(x_i) = 0$ for $x_i \in F(i)$, then there is an $l \geq i$ such that $\phi_{l,i}(x_i) = 0 \in F(l)$.

This is left as an exercise for the student. (Hint: By adding on appropriate elements and replacing x_i by $\phi_{i',i}(x_i)$, one can assume in $\bigoplus_j F(j)$ that $x_i = \sum_{i,j}(\phi_{i,j}(x_j) - x_j)$ where $i \geq j$ for each non-zero term in the sum.)

Now let $x' = \psi'_i(x'_i) \in M'$ and suppose $x'_i \mapsto x_i \in F(i)$ with $\psi_i(x_i) = 0 \in M$. Then, by (2), there is a $k \geq i$ such that $\phi_{k,i}(x_i) \to 0 \in F(k)$. Hence, $\phi_{k,i}(x_i) = 0$, so $\psi_i(x_i) = \psi_k(\phi_{i,i}(x_i)) = 0$. $\square$

Let F be a functor from a directed set I to the category of R-modules. A subset I' is said to be *cofinal* Note that a cofinal set is also directed. if for each $i \in I$, there is a $i' \in I'$ with $i' \geq i$. Let F' be the restriction of F to I'. Using the homomorphisms

$$\psi_{i'} : F'(i') = F(i) \to M = \lim_{\to} F \qquad i' \in I'$$

we get $\Psi : M' \lim_{\to} F' \to M$ making the appropriate diagrams commute, i.e., $\Psi \circ \psi'_i = \psi_{i'}$ for each $i' \in I'$.

PROPOSITION 11.10. *With the above notation, if I is directed and I' is cofinal in I, then Ψ is an isomorphism.*

PROOF. Ψ is an epimorphism. For, let $x = \psi_i(x_i) \in M$ with $x_i \in F(i)$. Let $i' \geq i$ with $i' \in I'$. Then,

$$\Psi(\psi'_{i'}(\phi_{i',i}(x_i)) = \psi_{i'}(\phi_{i',i}(x_i)) = x.$$

Ψ is a monomorphism. For, let $x' = \psi'_{i'}(x_{i'}) \in M'$ with $x_{i'} \in F(i')$, and suppose $\Psi(x') = 0$. Calculating as above shows that $\psi_{i'}(x_{i'}) = 0$, but this is the same as saying $\psi_{i'}(x_{i'}) = 0$. $\square$

Let M be an n-manifold, and consider the family $\mathcal{K}$ of compact subsets K of M ordered by inclusion. This is a directed set since if K, L are compact, so is $K \cup L$. If $K \subseteq L$, then $M - K \supseteq M - L$ so in cohomology we get an induced homomorphism

$$\rho^q_{L,K} : H^q(M, M - K; R) \to H^q(M, M - L; R).$$

In this way, $H^q(M, M - K; R)$ is a functor from $\mathcal{K}$ to the category of R-modules. We define the *cohomology with compact supports* by

$$H^q_c(M; R) = \lim_{\rightarrow} H^q(M, M - K; R).$$

This may also be defined in a slightly different way. Let

$$S^q_c(M; R) = \lim_{\rightarrow} S^q(M, M - K; R).$$

Note that if $K \subseteq L$, then we actually have

$$S^q(M, M - K; R) \subseteq S^q(M, M - L; R) \subset S^q(M; R)$$

and $S^q_c(M; R)$ is just the union of all the $S^q(M, M - K; R)$. It consists of all q-cochains f which *vanish outside some compact subset*, where the compact set would in general depend on the cochain. This explains the reason for the name 'cohomology with compact support'.

Define $\delta^q : S^q_c(M; R) \to S^{q+1}(M; R)$ is the obvious way. Then $S^*(M; R)$ is a cochain complex, and we may take its homology. Since direct limits on directed sets preserve exact sequences, it is easy to check that taking cocycles, coboundaries, and cohomology commutes with the direct limit. Hence, $H^q_c(M; R)$ is just the q-dimensional homology of the complex $S^*(M; R)$.

5. Proof of Poincaré Duality

Let M be an R-oriented n-manifold. For each compact subspace K of M, we may define a homomorphism

$$S_n(M, M - K; R) \otimes_R S^q(M, M - K; R) \to S_{n-q}(M; R)$$

by

$$c \otimes f \mapsto c \cap f.$$

This makes sense because if σ is a singular n-simplex $S_n(M - K; R)$, its front q-face $\sigma \circ [e_0, \ldots, e_q]$ is also in $S_n(M - K; R)$. Hence, if $f \in S^q(M, M-K; R)$; i.e., f is a q-cochain on M which vanishes on $S_1(M - K)$, then $\sigma \cap f = 0$. Hence, $c \cap f$ depends only on the class of c in $S_n(M, M - K; R) = S_n(M; R)/S_n(M - K; R)$.

Taking homology, we see get a sequence of homomorphisms

$$P_K^q : H^q(M, M - K; R) \to H_{n-1}(M; R)$$

defined by $a \mapsto \mu_K \cap a$, where μ_K is the fundamental class of K. It is not hard to check that if $K \subseteq L$, then

$$P_L^q \circ \rho_{L,K}^1 = P_K^q$$

so it follows that there is an induced homomorphism on the direct limit

$$P^q : H_c^q(M; R) \to H_{n-q}(M; R).$$

THEOREM 11.11. *Let M be an R-oriented n-manifold. The homomophisms*

$$P^q : H_c^q(M; R) \to H_{n-q}(M; R).$$

are isomorphisms.

Note that if M is itself compact, then $\{M\}$ is a cofinal subset of $\mathcal{K}$, and of course the direct limit that subset is just $H^q(M, M - M; R) = H^q(M; R)$. Hence, in this case $H^q(M; R) \cong H_c^q(M; R)$ and it is easy to check that P^q is just the homomophisms discussed previously. Hence, we get the statement of Poincaré Duality given previously for M compact.

PROOF. **Step 1**. Assume $M = \mathbf{R}^n$.

Let B be a closed ball in $\mathbf{R}^n$. Then as above since $R^n - B$ has S^{n-1} as a deformation retract,

$$H_q(\mathbf{R}^n, \mathbf{R}^n - B; R) = \begin{cases} R & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, an orientation for $\mathbf{R}^n$ implies a choice of generator $\mu_B \in H_n(\mathbf{R}^n, \mathbf{R}^n - B; R) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - B; R)$ (where $x \in B$). We claim the homomorphism

$$H^q(\mathbf{R}^n, \mathbf{R}^n - B; R) \to H_{n-q}(\mathbf{R}^n)$$

defined by $a \mapsto \mu_B \cap a$ is an isomorphism for $0 \le q \le n$. For, if $q \ne n$, then both sides are zero. For $q = n$, the homomorphism is just the evaluation morphism, and the result follows by the universal coefficient theorem

$$H^n(\mathbf{R}^n, \mathbf{R}^n - b; R) \cong \operatorname{Hom}(H_n(\mathbf{R}^n, \mathbf{R}^n - B), R) \cong \operatorname{Hom}_R(H_n(\mathbf{R}^n, \mathbf{R}^n - B; R), R).$$

since μ_B is a generator of $H_n(\mathbf{R}^n, \mathbf{R}^n - B; R)$ and both sides of the homomorphism are R. (Also, the Tor terms are zero.)

It follows that we have an isomorphism of the direct limit

$$\lim_{\rightarrow} H^q(\mathbf{R}^n, \mathbf{R}^n - B; R) \cong H_{n-q}(\mathbf{R}^n; R)$$

where the limit is taken over all closed balls B of $\mathbf{R}^n$. However, it is easy to see that for $\mathbf{R}^n$, the set of closed balls is cofinal, so we conclude that

$$P : H_c^q(\mathbf{R}^n; R) \to H_{n-1}; R)$$

is an isomorphsim.

Step 2. Suppose $M = U \cup V$ and that the theorem is true for U, V and $U \cap V$. We shall show that it is true for M.

We construct a Mayer–Vietoris sequence for cohomology with compact supports.

Let K, L be compact subsets of U, V respectively. Then there the relative Mayer–Vietoris sequence described previously in Section 1 yields for cohomology

$$\begin{aligned}\to H^q(M, M - K \cap L; R) \to H^q(M, M - K; R) \oplus H^q(M, M - L; R) \\ \to H^q(M, M - K \cup L; R) \to H^{q+1}(M, M - K \cup L; R) \to \dots\end{aligned}$$

Note that this is dual to the relative sequence described previously.

Let $C = M - U \cap V$. Then C is closed and contained in the interior of $M - k \cap L$, so by excision, we have a natural isomorphism

$$H^q(M, M - K \cap L; R) \cong H^q(U \cap V, U \cap V - K \cap L; R).$$

By similar excision arguments we have

$$\begin{aligned}H^q(M, M - K; R) \cong H^q(U, U - K; R) \\ H^q(M, M - L; R) \cong H^q(V, V - L; R)\end{aligned}$$

Now consider the following diagram

where the top sequence is obtained by replacing the cohomology groups in the above sequence by their isomorphs under excision, and the bottom sequence is the homology Mayer–Vietoris sequence for the triple $M = U \cup V, U, V$. The vertical arrows are the relative cap product morphisms discussed previously for each of the pairs $(U \cap V, U \cap V - K \cap L)$, $(U, U - K)$, $(V, V - L)$, and $(M, M - K \cup L)$.

LEMMA 11.12. *The above diagram commutes.*

We shall not try to prove this Lemma here. Suffice it to say that it involves a lot of diagram chasing and use of properites of the cap product. (See Massey for details.)

Assuming that the diagram commutes, we may take direct limits for the upper row. The set of all $K \cap L$ is cofinal in the set of all compact subsets of $U \cap V$, K and L are arbitrary compact subsets of U and V respectively, and the set of all $K \cup L$ is cofinal in the set of all compact subsets of $M = U \cup V$. If follows that we get a diagram

where the upper line is what we will call the Mayer-Vietoris sequence for cohomology with compact supports. The upper line remains exact since direct limits preserve exactness. We may now derive the desired result by means of the five lemma.

Step 3. Let I be a *linearly ordered set* and let M be the union of an ascending chain of open subsets U_i indexed by $i \in I$, i.e., assume

$$i \leq j \Rightarrow U_i \subseteq U_j.$$

Assume the theorem is true for each U_i. We shall prove the theorem for M.

Note first that $H_r(U_i; R)$ is a functor on the directed set of U_i.

LEMMA 11.13. $\varinjlim H_r(U_i; R) = H_r(M; R)$.

PROOF. The support of any cycle c in $S_r(M; R)$ is compact and so must be contained in one of the U_i. Using this, it is not hard to check the above lemma. □

Now let K be a compact subset of some U_i. Excision of $M - U_i$ yields an isomophism

$$H^q(M, M - K; R) \cong H^q(U, U - K; R).$$

It follows that the *inverses* of these isomorphisms for K compact in U_i yield an isomorphism of direct limits

$$\lim_{\rightarrow} H^q(U_i, U_i - K; R) \to \lim_{\rightarrow} H^q(M, M - K; R)$$

where both limits are taken over the directed set of compact subset of U_i. The left hand side by definition $H^q_c(U_i; R)$. On the right, since we only have a subfamily (not even cofinal) of the set of compact subsets of M, the best we have is a homomorphism

$$\lim_{\rightarrow} H^q(M, M - K; R) \to H^q_c(M; R).$$

Putting this together, we get the upper row of the following diagram

$$\begin{tikzcd} H^q_c(U_i; R) \arrow[r] \arrow[d] & H^q_c(M; R) \arrow[d] \\ H_{n-1}(U_i; R) \arrow[r] & H^q_c(M; R) \end{tikzcd}$$

where the vertical arrows are the Poincaré Duality homomorphisms. Moreover, it is possible to see with some effort that this diagram commutes.

Now take the limit with respect to the U_i. On the bottom, we get an isomorphism, as mentioned in the above Lemma. The vertical arrow on the left is an isomorphism because by assumption it is a direct limit of isomorphisms. (Think about that!). Hence, it suffices to see that the arrow on top is an isomorphism. To see this use the general principle that *direct limits commute*. (We leave it to the student to state that precisely and to prove it.) On the left, we are first fixing i, then taking the limit with respect to $K \subseteq U_i$ and then taking the limit with respect to i. We could just as well fix a K and consider only those $U_i \supseteq K$. (The set of such is certainly cofinal in the set of all U_i.) Since, as noted above, all the $H^q(U_i, U_i - K; R) \to H^q(M, M - K; R)$ are isomorphisms, it follows that in the limit over all such i, we get an isomorphism. Now take the limit with respect to the set of all compact K in M. The limit is again an isomorphism.

Step 4. We establish the theorem for any open set M in $\mathbf{R}^n$.

First note that if M is convex, then it is homeomorphic to $\mathbf{R}^n$, so the theorem is true by Step 1. More generally, we can cover M by a family of open convex subsets $V_1, V_2, \ldots$. Define

$$\begin{aligned} U_1 &= V_1 \\ U_2 &= V_1 \cup V_2 \\ &\vdots \\ U_i &= U_{n-1} \cup V_n \\ &\vdots \end{aligned}$$

The theorem is true for each U_i by Step 2 and induction. Hence, the theorem is true for M by step 3.

Step 5. M may be covered by a countable collection of Euclidean neighborhoods.

Use the same argument as in Step 4. Note that this takes care of all the usual interesting manifolds.

Step 6. The general case.

We need a slight diversion here about transfinite induction.

Zorn's Lemma.

Let I be a *non* partially ordered set. We say that I is inductively ordered if any linearly ordered subset has an upper bound. Then *Zorn's Lemma* asserts that I has a maximal element, i.e., an element m such that there is no element $i \in I$ with $i > m$. Note that this doesn't say that every element of I is bounded by m. Such an element would be unique and be called the *greatest* element of I. There may be many maximal elements.

Zorn's Lemma is not really a lemma. It can be shown to be logically equivalent to several other assertions in set theory. The most notable of these are the *axiom of choice* and *well ordering principle*. You can learn about this material by studying some logic. Some mathematicians consider the use of these assertions questionable. Indeed, Paul Cohen showed that the axiom of choice in a suitable sense is independent of the other axioms of set theory.

However, we shall use Zorn's Lemma, blinking only slightly in the process.

To complete Step 6, Let $\mathcal{U}$ be the collection of all connectged open sets of M for which the theorem is true, ordered by inclusion. Since the theorem is true for every Euclidean neighborhood of M, the set $\mathcal{U}$ is non-empty. It is also inductively ordered by Step 3. Hence, by Zorn's Lemma, there is a maximal connect open set U of M for which the theorem is true. We claim that $U = M$. Otherwise, let $x \in \partial(M - U)$

and choose an open Euclidean neighborhood V of x. By Step 2, the theorem is true for $U \cup V$, and this contradicts the maximality of U. $\square$